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Anomalous diffusion and scaling behavior of dynamically perturbed one-dimensional disordered quantum systems

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Abstract

Coherent oscillatory perturbations enhance the localization length of one-dimensional quantum disordered systems to a numerically undetectable level and result in an anomalous diffusion. The transition to the normal diffusion occurs continuously with the perturbation strength and/or the number of frequency components of the oscillatory perturbation. The corresponding space(x)–time(t) distribution function $P(x,t)$ reduces to the unified scaling form $P(x,t) \sim \exp[-\mathrm{const} \times (x/t^{\alpha/2})^{\beta}]$, which contains the localization and the normal diffusion as two extreme cases and interpolates the two limits in the general case. © 1998 Published by Elsevier Science B.V.

The localization phenomena in one-dimensional disordered quantum systems (1-DDS) have been extensively studied since several decades ago [1,2]. It is well known that almost all the eigenstates are localized in the presence of any disorder [3]. The localization is regarded as a result of coherent backscattering by irregularly distributed scatterers [2]. There are generally two length scales in the disordered quantum systems; one is the mean free path ℓ and the other is the localization length L [4]. The apparently diffusive behavior of the wave packet, which is initially localized on a lattice site, can be observed until the wave packet length reaches the second length scale L.

However, the existence of such a diffusive regime has not been established in the ordinary 1-DDS, because these two lengths are almost of the same scales [1,2]. If the spatial dimension is more than one, the localization length L becomes much longer than ℓ or maybe infinite, and an anomalous diffusive motion obeying a power law t^{α} $(0< \alpha \leqslant 1)$ becomes observable [5]. It is not, however, easy to confirm numerically whether or not the presence of such an anomalous (or maybe normal) diffusion regime is intrinsic.

If the localization mechanism is destroyed by some mechanism, we may expect that the wave packet will exhibit a diffusive motion rather than a ballistic motion, but the nature of the quantum diffusion which is supposed to appear in the delocalization regime has not been fully understood. A simple and realistic way to destroy the localization mechanism is to introduce a coupling with some other dynamical degrees of freedom. In fact, some class of classically chaotic quantum systems, which exhibit the Anderson localization, may recover the classical normal diffusion by introducing a coupling with other dynamical degrees of freedom [6]. In actual situations, electrons being scattered by irregularly distributed impurities are also

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perturbed by the oscillations due to lattice vibrations. Such an oscillatory perturbation will destroy the coherence yielding the localization phenomenon and may result in a stationary conduction of electrons. Mott first pointed out that the oscillatory perturbation applied to the localized system may induce the hopping of an electron over a distance much longer than L [7]. However, the approach is based upon the perturbation theory, and it can not predict whether the longer time behavior is an intrinsic quantum diffusion or not [7,8]. On the other hand, the effect of oscillatory perturbation on the disordered quantum system has been studied also for the dynamical persistent current in disordered mesoscopic rings and cylinders [9] and for the Zener tunneling transition [10]. These treatments are also based upon the perturbation theory.

The aim of this present paper is to investigate how the localization phenomenon is influenced by introducing a coupling with other dynamical degrees of freedom, which is simulated by *classical oscillatory perturbations*. In preliminary reports we showed that the localization length is much enhanced by adding a single-frequency oscillatory perturbation [12]. In the present paper, our concern is how the number of degrees of freedom together with the strength of the coupling changes the quantum dynamics of wave packet which is localized without the dynamical perturbation. As is expected, the localization length is much enhanced to the level undetectable by numerical experiments. The main results are that an anomalous diffusion emerges and the normal diffusion in the rigorous sense is achieved in a certain limit, and that the spatio-temporal evolution of the distribution function associated with the anomalous diffusion obeys a scaling rule which contains the localization type and the normal diffusion type distributions as the two opposite limits.

The model treated in this paper is a 1-DDS with nearest-neighbor interaction, which is perturbed by periodically oscillating forces:

$$H(t) = \sum_{n=1}^{N} |n\rangle (V_0(n) + V_1(n,t)) \langle n| + \sum_{n=1}^{N} (|n\rangle\langle n+1| + |n+1\rangle\langle n|), \qquad (1)$$

$$V_1(n,t) = V_1(n) \frac{\epsilon}{\sqrt{M}} \sum_{j=1}^{M} \cos(\omega_j t). \qquad (2)$$

The basis set $\{|n\rangle\}$ is an orthonormalized one representing lattice sites, and $V_0(n)$ and $V_1(n)$ are the on-site energy of an electron at site n, and $V_0(n)$ varies randomly in the range $[-W, W]$ from site to site. The oscillatory perturbation is polychromatic and is composed of M different frequency components (colors). We suppose all the amplitudes of the frequency components are the same, and so the long-time average of the squared perturbation amplitude is given by $\langle V_1(n,t)^2\rangle_t / V_1(n)^2 = \frac{1}{2}\epsilon^2$ ($\langle\ldots\rangle_t$ indicates the long-time average), and thus ϵ characterizes the perturbation strength. The frequencies $\{\omega_i\}$ are so chosen as mutually incommensurate, which are typically given as $\omega_1 = 1$, $\omega_2 = 1 + \sqrt{2/25}$, $\omega_3 = 1 + \sqrt{3/25}, \ldots$, and so on. A similar model with oscillatory perturbation applied to the off-diagonal part of tight-binding model has been used in order to investigate the diffusion property [11].

There are various ways of choosing the interaction potential: one possibility is to take $V_1(n)$ as a random function of n, and another is to choose $V_1(n)$ as a systematic function of n. In the present paper, we mainly report the former case, $V_1(n) = V_0(n)$, namely the on-site random potential is perturbed by the same oscillatory force.

The time-dependent wave packet, which is formally represented by

$$\Psi(t) = \exp\left(-\mathrm{i} \int_0^t [2\cosh(\partial/\partial x) + V_0(x) + V_1(x,s)]\, \mathrm{d}s \right) \Psi(0) \qquad (3)$$

($\hbar = 1$), can be computed numerically by combining the higher-order symplectic integrator (SI) with the fast-Fourier transform (FFT) [13]. Since the FFT is done almost exactly by choosing the number of lattice sites $N = 2^{\mathrm{integer}}$ under the periodic boundary condition, $\Psi(t)$ can be computed at a very high precision by the FFT–SI scheme. We only consider time evolution in which the wave front of the packet does not reach the boundary in order to avoid influencing the boundary condition and finite size effect. We used the

6th order SI with the site number $N = 1024$ and the size of the SI integration step $\Delta t = 0.05$.

To observe the temporal behavior of the wave function, we show the time dependence of the mean-square displacement (MSD) of the wave packet;

$$\xi(t)^2 \equiv \langle\langle\Psi(t)|(\hat{n} - \langle\hat{n}\rangle)^2|\Psi(t)\rangle\rangle_\Omega, \tag{4}$$

where $\hat{n} \equiv \sum_{n=1}^{N} n|n\rangle\langle n|$ is the position operator, the angular brackets, $\langle\ldots\rangle$ and $\langle\ldots\rangle_\Omega$, means quantum and sample averages (averaged over 20–50 samples of random potentials), respectively. We examine how the nature of the Anderson localization changes as the number of independent frequencies M and/or the intensity of perturbation ϵ are increased.

In case of $M = 1$, the diffusive behavior is observed within the time scale t_L on which the unperturbed system becomes localized [12], but such a temporal diffusion is suppressed on a longer time scale, and the MSD evidently saturates at a level much larger than the localization length L of the unperturbed system. For $M \geqslant 2$, however, the presence of the localization length can no longer be recognized on the time scale accessible by computer simulation [1]. But the MSD does not exhibit the normal diffusion over a time scale much longer than t_L. To elucidate the nature of the temporal evolution process of MSD, we show in Fig. 1 the log–log plot of MSD as functions of time.

Except for the initial regime of time evolution, where the transient effect still remains, all the data except for $M = 1$ align along straight lines with tangents less than 1, and so we may conclude that the MSD obeys a well-defined subdiffusion law,

$$\xi(t)^2 = Dt^\alpha \quad (0 < \alpha < 1). \tag{5}$$

The exponent α estimated from the numerical data is plotted in Fig. 2a. We find that the exponent α gradually increases and approaches 1, which means the onset of normal diffusion. This fact seems to imply that at finite ϵ the normal diffusion in the mathematically rigorous sense is achieved only in the limit of $M \to \infty$.

Next, we investigate in detail how the transition from the subdiffusion to the normal diffusion takes

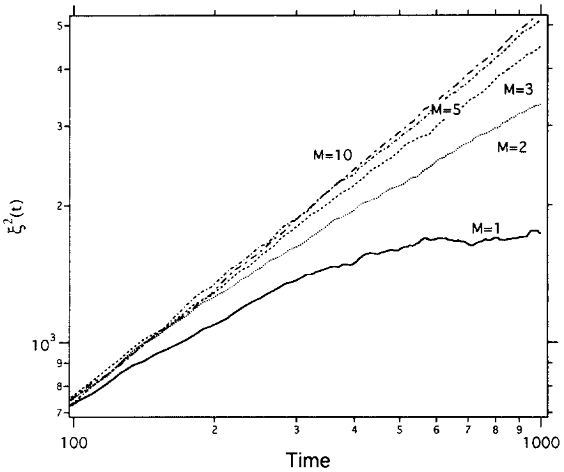

Fig. 1. Log–log plot of MSD versus time for $M = 1, 2, 3, 5$ and 9 at the perturbation strength $\epsilon = 0.5$.

place as the perturbation strength ϵ is varied continuously. We show in Fig. 2b how the exponent α changes as the perturbation strength ϵ increases at some representative numbers of colors, i.e., $M = 2$, 5 and 10. In all cases the exponent α increases steeply as ϵ increases, and eventually approaches to unity. As the number M increases, the approach to $\alpha = 1.0$ with an increase in ϵ becomes more steep. However, the transition from the subdiffusion to the normal diffusion is continuous, and there seems to be no drastic change. Further, even though the number of colors M is large enough, the diffusion process obeys a subdiffusion law in a very weak regime of ϵ. This is the essentially different nature from the stochastic perturbation, which induces a normal diffusion however small the perturbation strength may be.

We conclude that the Anderson localization seems to be destroyed if the number of colors is more than 1. Even though the localization was not destroyed, the localization length is extremely enhanced to the level undetectable by numerical simulations, and the MSD increases obeying a well-defined subdiffusion law until the numerically undetectable localization length (if it were) is reached. We call such a phenomenon the *dynamical delocalization* [12]. A basic question arising here is whether or not there is any transition from the localized state to the seemingly delocalized state in an extremely small regime of ϵ. This is a very difficult question which is not easy to answer from the numerical investigations.

[1] In the case of kicked rotors, the coupling with the oscillatory perturbation is roughly equivalent to increase the spatial dimension and transition to the delocalization is observed [14].

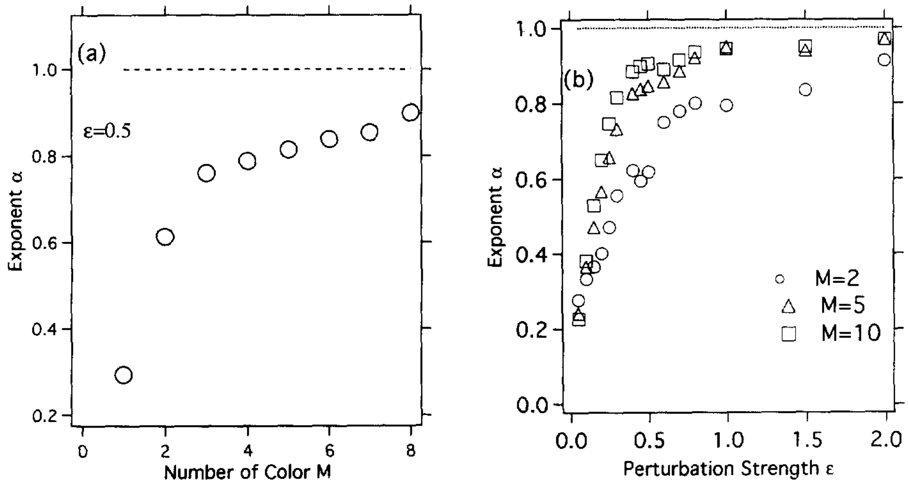

Fig. 2. (a) M and (b) ϵ dependence of the exponent α, where $\epsilon = 0.5$ in (a) and $M = 2, 5, 10$ in (b).

The MSD, which obeys the subdiffusion law for $M \geqslant 2$ approaches rapidly but continuously to the normal diffusion law. In the latter regime, we can expect that the distribution function

$$P(n,t) \equiv \langle |\Psi(n,t)|^2 \rangle_\Omega \tag{6}$$

evolves in the Gaussian form $P(n,t) \propto \exp[-(n-n_0)^2/2\xi(t)^2]$. On the other hand, the localization occurs at least in the case of $M \leqslant 1$, in which the wave packet decays in a linear-exponential form, $P(n,t) \propto \exp[-|n-n_0|/\xi(t)]$. In both limits, it is evident that the distribution function obeys the scaling rule

$$P(n,t_2) = \frac{\xi(t_1)}{\xi(t_2)} P\left(\frac{\xi(t_1)}{\xi(t_2)} n, t_1\right)$$

at two arbitrary times t_1 and t_2 in the stage of the stationary time evolution (i.e., $t > t_L$). We may expect that such a scaling rule can be extended also to the subdiffusion regime. We first examine whether our expectation is valid or not. To do this examination, we supposed the distribution $P(n,t^*)$ at time $t^*(=700)$ as a standard distribution, and examined whether the distribution function at t^* constructed from the one at an arbitrary t by setting $t_2 = t^*$ and $t_1 = t$ according to the scaling ansatz

$$P_t(n,t^*) = \frac{\xi(t)}{\xi(t^*)} P\left(\frac{\xi(t)}{\xi(t^*)} n, t\right).$$

We observed that such a scaling rule is just satisfied in a wide range of the parameters ϵ and M. In Fig. 3a we show some typical examples of the semi-log plots of the $P_t(n,t^*)$ obtained at several ts in the cases of $M = 2, 3$, and 10 at $\epsilon = 0.5$, where the smoothed distribution functions are displayed. Except for the sample at $t = 300$, which is plotted as a typical example in the early stage of time evolution, the $P_t(n,t^*)$s coincide with each other very well, and thus we have to conclude that $P(n,t)$ is represented by the scaled form,

$$P(n,t) = \xi(t)^{-1} P_{\text{sc}}\left(\frac{n}{\xi(t)}\right). \tag{7}$$

We investigate further details of the functional form of $P_{\text{sc}}(x)$. Let $g(x)$ be the exponent function defined by $P_{\text{sc}}(x) = e^{-g(x)}$. Then in the normal-diffusion limit and in the localization limit, it is $g(x) = x^2/2 + \text{const}$ and $g(x) = |x| + \text{const}$, respectively. We thus may expect that the exponential part $g(x)$ shows in general the power-law dependence $g(x) - g_0 \propto |x|^\beta$, which interpolates the two limiting situations, where the exponent β will take a fractional value between 1 and 2 and g_0 is a certain constant. The unknown parameters g_0 and β are determined so that $Q(x,g_0) \equiv \log[-\log P_{\text{sc}}(x) + g_0]$ may fit best to the linear function of $\log|x|$ i.e., $\beta \log|x| + \text{const}$. In Fig. 3b we show some examples of the best-fitted $Q(x,g_0)$. The fitting is very nice in the significant range of x, which strongly indicates the validity of the power-law dependence. Existence of such a power-law dependence is verified for various combinations of the parameters

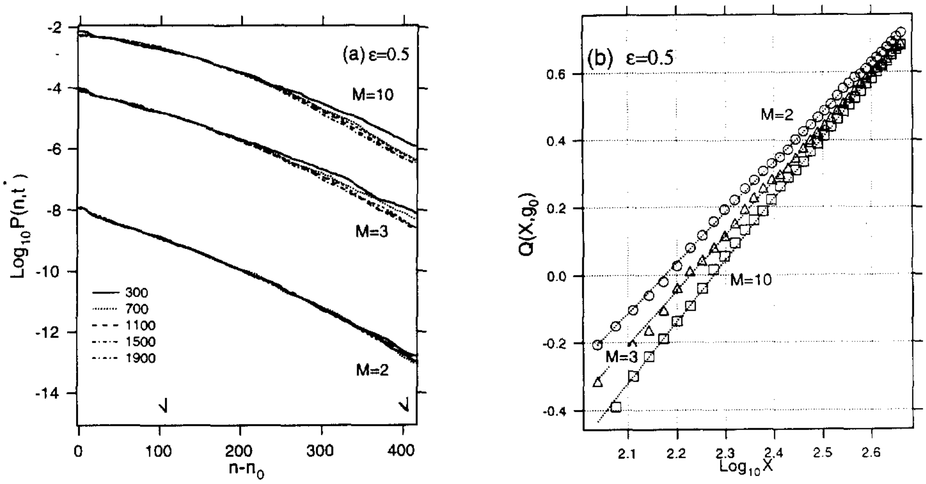

Fig. 3. (a) Typical example of the scaling test of the distribution function, where $P_t(n, t^* = 700)$ constructed at various t are superposed. The region bounded by the two arrows indicates the range used for the fitting of (b). To avoid overlapping the data we added -6.0 (-2.0) to the raw value in $M = 2$ $(M = 3)$ for a convenience. (b) Some examples of $Q(x, g_0)$ fitted best to the linear function.

ϵ and M. In conclusion, the scaled form of the distribution function is given by the "stretched" Gaussian distribution,

$$P(n,t) \propto \exp\left[-\text{const} \times \left(\frac{|n|}{t^{\alpha/2}}\right)^{\beta}\right], \tag{8}$$

except for the range very close to the center of the distribution. Thus the distribution function is specified by the two exponents, i.e. α characterizing the temporal growth of the wave packet, and β characterizing its spatial decay. Our scaled distribution function is of a unified form, which contains the two extreme limits, i.e. the localization $(\alpha = 0, \beta \doteq 1)$ and the normal-diffusion $(\alpha = 1, \beta = 2)$ as special cases and in general interpolates them.

The two exponents are not independent and should be connected at least in the normal diffusion limit and in the localization limit. The question arising here is whether or not there exists any correlation between the two exponents in the intermediate regime. Fig. 4 shows the plot of (α, β) obtained for various combinations of the two independent parameters ϵ and M. The precise determination of β is more difficult than α, and the former is accompanied by some amount of error, but it is evident that the two exponents are strongly correlated in the subdiffusion regime. It is quite an interesting question whether (α, β) is on a single unique curve or not. To answer this question, a more rigorous numerical test is desired.

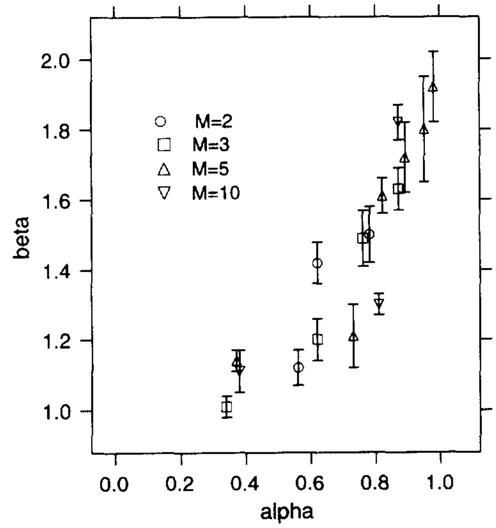

Fig. 4. Plot of (α, β) obtained at various combinations of ϵ and M. Note that $(\alpha, \beta) = (0, 1)$ is the localization limit and $(1, 2)$ is the normal diffusion limit.

The features mentioned above seem to be insensitive to the coupling scheme of the oscillatory perturbation. In fact we observed the same features for the systematic coupling $V_1(n) = n$, which correspond to the AC electric field perturbation [15] (in Ref. [16], the time-dependent property of electron is investigated in a semiconductor superlattice with disorder driven by an AC electric field, as a realistic model).

In conclusion we have shown that the oscillatory perturbation containing more than one frequency component, enormously enhances the one-dimensional localization length and an anomalous diffusion that obeys a unified scaling rule is achieved.

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