Delocalization due to Time-Dependent Perturbation in One-Dimensional Disordered Systems

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- 1. Quantum diffusion in stochastic fluctuating potential
- 2. Scaling of dynamical localization
- 3. Maryland Transform
- 4. Self-Consistent Mean-Field Theory

Localization-delocalization transition (LDT)

- 5. Critical dynamics at LDT
- 6. Scaling of the dynamics at LDT
- 7. Critical curves of LDT

Quantum diffusion in 1DDS (lattice model)

$$i\hbar\frac{\partial\phi(n,t)}{\partial t} = \phi(n+1,t) + \phi(n-1,t) + V(n,t)\phi(n,t),$$

•the mean square displacement (MSD),

$$m_2(t) = \sum_n (n-n_0)^2 \left\langle |\phi(n,t)|^2 \right\rangle \sim t^\alpha$$

$$\text{diffusion exponent} \begin{cases} \alpha = 0 & \text{localization} \\ 0 < \alpha < 1 & \text{subdiffusion} \\ \alpha = 2/3 & \text{MIT in 3DDS?} \\ \alpha = 1 & \text{normaldiffusion} \end{cases}$$

•Dynamical localization length (DLL): $\xi_{msd} = \sqrt{m_2(t\to\infty)}$

Stochastic fluctuating cases 1 (lattice model)

$$H(t) = \sum_{n=1}^{N} |n\rangle V(n,t)\langle n| - J\sum_{n}^{N}(|n\rangle\langle n+1| + |n+1\rangle\langle n|),$$

•Gaussian white noise (Haken te.al 1972, Ovchinikov 1974,Madhukar et al 1977)

$$C(t_1 - t_0) = \langle V(n_0,t_0)V(n_1,t_1)\rangle = W^2\delta_{n_0,n_1}\delta(t_1-t_0),$$
$$D = \lim_{t\to\infty}\frac{< n^2 >}{t} = \frac{J^2}{W^2},$$

•Heat bath model (Sumi 1974)
Dynamical CPA
•Two-states colored noise (Inaba 1981, Ezaki 1991)

Stochastic fluctuating cases 2 (lattice model)

•Gaussian Markovian noise (Colored noise)
(Kitahara et al 1979, Sato et al 1984, Palenberg et al 2000, Amir et al 2009, Moix et al 2013, Errico et al 2013)

$$
\begin{aligned}
C(t_1 - t_0) &= W^2 \delta_{n_0, n_1} e^{-\gamma |t_1 - t_0|}, \\
D &= \begin{cases} J^2 \left[\frac{\gamma}{W^2} + \frac{\gamma}{\gamma^2 + J^2} \right] & W >> J, \\ J^2 \frac{\gamma}{W^2} \left[1 + \frac{3J^2}{\gamma^2} \right] & W << J, \end{cases}
\end{aligned}
$$

•Localization + Colored noise (Yamada et al 1999, Gopalakrishnan et al 2017)

$$V(n,t) = \epsilon_n + v(n,t)$$

ϵ_n: random site-energy, $v(n,t)$:colored noise

Stochastic fluctuating cases 3 (continuous model)

$$\begin{aligned} i\hbar\frac{\partial^2\phi(x,t)}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2} + V(x,t)\phi(x,t), \\ \phi(x,t=0) &= \frac{1}{(\sqrt{2\pi}\sigma)^{1/2}}\mathrm{e}^{-x^2/4\sigma^2} \end{aligned}$$

•Gaussian white noise (Jayannavar and Kumar 1982)

$$\begin{aligned} \langle V(x_0,t_0)V(x_1,t_1)\rangle &= W^2 g(x_0-x_1)\delta(t_1-t_0) \\ g(y) &= \frac{1}{\sqrt{2\pi}\alpha}\mathrm{e}^{-y^2/2\alpha^2}, \\ \langle x^2\rangle &= \sigma^2 + \frac{\hbar^2}{4m^2\sigma^2}t^2 + \frac{1}{3\sqrt{2\pi}}\frac{W^2}{m^2\alpha^2}t^3 \end{aligned}$$

Note: in a classical limit, $\hbar \to 0$,

$$\langle x^2\rangle = \sigma^2 + \frac{1}{3\sqrt{2\pi}}\frac{W^2}{m^2\alpha^2}t^3$$

Stochastic fluctuating cases 4 (continuous model)

•Gaussian colored noise (Heinrichs 1992, Jayannavar 1993)

$$\begin{aligned} \langle V(x_0,t_0)V(x_1,t_1)\rangle &= W^2 g(x_0-x_1)h(t_1-t_0), \\ h(t_1-t_0) &= \frac{1}{2\tau}\mathrm{e}^{-|t_0-t_1|/\tau} \\ g(y) &\sim \mathrm{e}^{-\alpha y^4} \end{aligned}$$

$$\left\langle x^2\right\rangle \sim t^2, \left\langle x^{2n}\right\rangle \sim t^{5n/2} : even, \sim t^{(5n-1)/2} : odd$$

Generally, for $g(y) = \mathrm{e}^{-\alpha y^{2m}}$ $(m > 1)$,

$$\left\langle x^{2n}\right\rangle \sim \begin{cases} t^{2n} & (n \le m) \\ t^{\frac{2m+1}{m}n} & (n > m) \end{cases},$$

•c.f. Debye Wallar factor, motional narrowing, variable-range hopping

Stochastic fluctuating cases 5 (continuous model)

•Quasi-periodic space-time perturbation (Krivoiapov et al 2012)

$$V(x,t) = \frac{1}{\sqrt{N}} \sum_{m=-N}^{N} A_m e^{i(k_m x - \omega_m t)},$$

$$\begin{aligned} A_{-m} &= A_m^*, \\ \langle A_m \rangle &= \langle A_m A_n \rangle = 0, \\ \langle A_m A_n^* \rangle &= \sigma^2 \delta_{n,m} \\ P(k,\omega) &: i.i.d. \end{aligned}$$

$$\langle x^2 \rangle \sim t^{12/5}, \langle p^2 \rangle \sim t^{2/5}$$

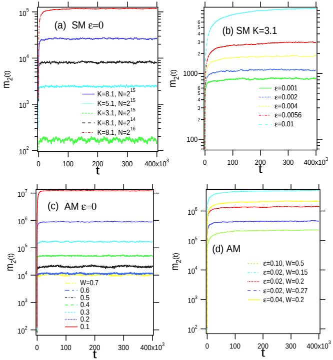
DL in coherently perturbed kicked 1DDS

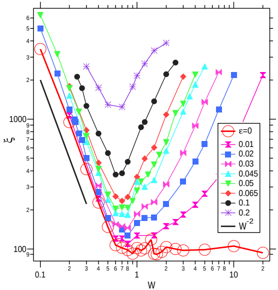
- Tightly Binding Model

$$
\begin{aligned}
H(t) &= \sum_{n=1}^{N} |n\rangle V(n)(1+f(t))\langle n|\delta_t - t\sum_{n}^{N}(|n\rangle\langle n+1| + |n+1\rangle\langle n|), \\
\delta_t &= \sum_{m} \delta(t-m), f(t) = \frac{\epsilon}{\sqrt{M}}\sum_{j=1}^{M}\cos(\omega_j t + \theta_j).
\end{aligned}
$$

- $V(n)$: spatial disorder (disorder strength W)
- ϵ: perturbation strength($\epsilon < W$)
- ω_i: incomensurate frequencies of order $O(1)$
- M: number of colors

•dynamical localization length: $\xi = \sqrt{m_2(\infty)} (\hbar = 1/8, \ \tau = 1)$



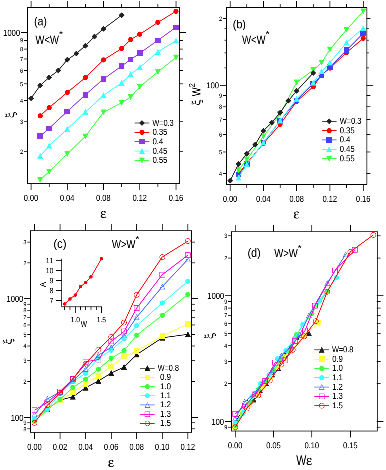


dynamical localization length

Characteristic disorder strength

$$\xi_0(\epsilon = 0) \simeq \begin{cases} c_0 W^{-2} & (W < W^*) \\ \xi_0^* & (W > W^*) \end{cases}, W^* \simeq \frac{2\pi}{\tau}\hbar = 0.78.$$

•Scaling of the DLL

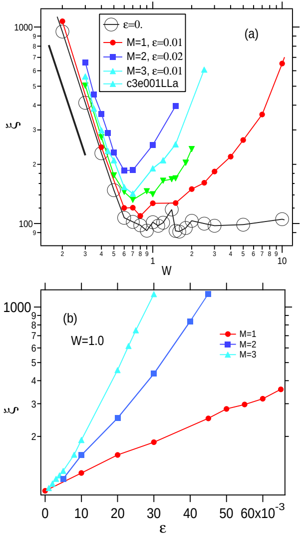


$$\xi(\epsilon, W) \simeq$$

$$\begin{cases} c_0 W^{-2} exp\{c_1 \epsilon\} & (W < W^*) \\ \xi_0^* exp\{c_2 W \epsilon\} & (W > W^*), \end{cases}$$

Scaling of DLL 3 (AM, $M \geq 2$)

•Scaling of the DLL

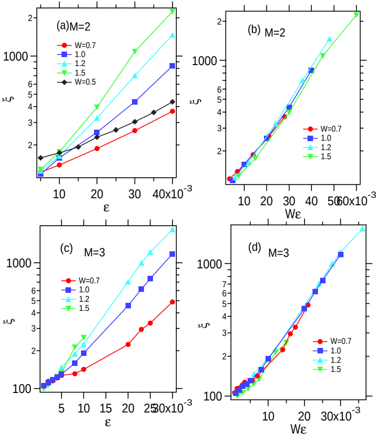


$$\xi(\epsilon, W) \simeq W^{-2} exp\{c_1 \epsilon\} \ \epsilon \ll 1$$

But, it grows faster than exponential with increasing ϵ !

Scaling of DLL 4 (AM, $M \geq 2$)

•Scaling of the DLL



$$\xi(\epsilon, W) \simeq$$

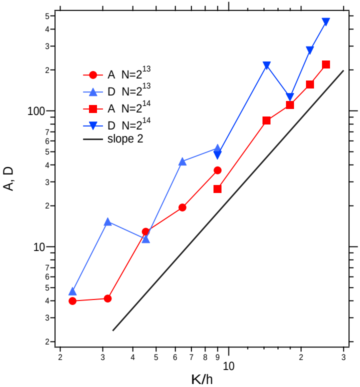
$$\begin{cases} c_0 W^{-2} exp\{c_1 \epsilon\} & (W < W^*) \\ \xi_0^* exp\{c_2 W \epsilon\} & (W > W^*), \end{cases}$$

Scaling of DLL 5 (SM,$M = 1$)

- Quantum standard map(SM)

$$H(\hat{p}, \hat{q}, t) = \frac{\hat{p}^2}{2} + K \cos \hat{q} \{1 + \epsilon \cos(\omega_1 t + \varphi_0)\} \delta_t,$$

$$H_{aut} = \frac{\hat{p}^2}{2} + \omega_1 \hat{J} + K \cos \hat{q} \left\{1 + \epsilon \cos \hat{\phi}\right\} \delta_t,$$



$(K/\hbar)-$dependence of DLL:

$$p_\xi = D\exp(A\epsilon)$$

$$p_\xi \propto \left(\frac{K}{\hbar}\right)^2 \exp\left[\text{const.}\epsilon \left(\frac{K}{\hbar}\right)^2\right]$$

in the quantum regime.

$$\epsilon << \hbar^2$$

Maryland Transform 1(AM,$M = 1$)

• one-step evolution operator when the monochromatic dynamical perturbation is taken into account by the $J-$oscillator in an autonomous way.

$$U_{tot} = \mathrm{e}^{-i\omega_1 \hat{J}/\hbar}\mathrm{e}^{-iT(\hat{p})/\hbar}\mathrm{e}^{-iV(\hat{q})(1+\epsilon\cos\hat{\phi})/\hbar}.$$

Maryland transform:

$$\hat{U}_{tot}|u_1\rangle = \mathrm{e}^{-i\gamma}|u_1\rangle$$

⇒Tight-binding form

• Generalization to the cases $M \geq 2$ is easy

Maryland Transform 2(AM, $M = 1$)

•This eigenvalue problem can be mapped into the tight-binding form for the 2DDS by the Maryland transform,

$$\hat{U}_{tot}|u_1\rangle = \mathrm{e}^{-i\hat{A}}\mathrm{e}^{-i\hat{B}}\mathrm{e}^{-i\hat{C}}|u_1\rangle = \mathrm{e}^{-i\gamma}|u_1\rangle$$

$$\begin{aligned} \hat{A} &= (Wv(\hat{q}) + \omega_1 \hat{J})\tau/\hbar, \\ \hat{B} &= \epsilon W v(\hat{q})(\cos\phi)\tau/\hbar, \\ \hat{C} &= 2\cos(\hat{p}/\hbar)/\hbar \end{aligned}$$

$$\left(\tan\left[\frac{(\hat{W}(\hat{q}) + \omega_1 \hat{J} - \gamma)\tau}{2}\right] + \hat{t}(\hat{p}, \hat{q}, \phi)\right)|\overline{u}\rangle = 0.$$

•$u(n,j) = (\langle n| \otimes \langle j|)|\overline{u}\rangle$ ($|j\rangle$ is eigenstate of the $\hat{J}$)

$$\tan\left[\frac{Wv_n + j\omega_1\hbar}{2\hbar}\tau - \frac{\gamma}{2}\right]u(n,j) + \sum_{n',j'}\langle n, j|\hat{t}|n', j'\rangle u(n', j') = 0,$$

Maryland Transform 3(AM,$M = 1$)

$$
\begin{aligned}
\langle n, j|\hat{t}|n', j'\rangle &= \left\langle n, j \left| i\frac{e^{-i\hat{B}} - e^{i\hat{C}}}{e^{-i\hat{B}} + e^{i\hat{C}}} \right| n', j' \right\rangle \\
&= \left\langle n, j \left| i\frac{e^{-i\epsilon W v(\hat{q}) \cos\phi\tau/\hbar} - e^{i2\cos(\hat{p}/\hbar)\tau/\hbar}}{e^{-i\epsilon W v(\hat{q}) \cos\phi\tau/\hbar} + e^{i2\cos(\hat{p}/\hbar)\tau/\hbar}} \right| n', j' \right\rangle .
\end{aligned}
$$

$\epsilon = 0$ as the simplest case of the hopping term

$$
\langle n, j|\hat{t}|n', j'\rangle = \frac{1}{2\pi}\int_0^{2\pi} dp \mathrm{e}^{i(n-n')p} \tan\left[\frac{\cos(p/\hbar)}{\hbar}\tau\right],
$$

where $p = 2\pi\hbar k/N$.

- In the small τ limit , $\tan(x) \simeq x$.

$$
\tau W/2\hbar \sim \pi \to W^* \simeq \frac{2\pi}{\tau}\hbar,
$$

- In $\epsilon = 0$ the relationship between AM and SM

$$
\frac{W}{\hbar} \Leftrightarrow \frac{\hbar}{K}.
$$

Self-Consistent Mean-Field Theory 1 (self-consistent Eq.)

- $D_\mu(\omega)$: dynamical diffusion constant in the μ direction ($\mu = 1, 2$)

$$\boxed{\frac{D_\mu(\omega)}{D_\mu^{(0)}} = 1 - \frac{1}{\pi\rho}\frac{D_\mu(\omega)}{D_\mu^{(0)}}\sum_{q_1,q_2}\frac{1}{-i\omega + \sum_{\nu=1}^{2} D_\nu(\omega) q_\nu^2}}$$

$$\xi_\mu(\omega)^2 = D_\mu(\omega)/(-i\omega),$$

- In the limit of $\omega \to 0$, $\xi_\mu(\omega) \to \xi(\omega = 0)$ localization length(LL)

$$\frac{\xi_\mu(\omega)^2}{\ell_\mu^2}(-i\omega t_\mu) = 1 - \frac{\xi_\mu(\omega)^2 t_\mu}{\ell_\mu^2}\frac{1}{\xi_1(\omega)\xi_2(\omega)}\Xi\left[\frac{\xi_1(\omega)}{\ell_1}, \frac{\xi_2(\omega)}{\ell_2}\right],$$

$t_\mu = \ell_\mu^2/D_\mu^{(0)}$: the localization time $\to$ $\boxed{t_2 = t_1 = \ell_1}$
ℓ_μ: characteristic length of the integration

Self-Consistent Mean-Field Theory 2

$$\Xi\left[\frac{\xi_1(\omega)}{\ell_1}, \frac{\xi_2(\omega)}{\ell_2}\right] = \tilde{c}\int_0^{\xi_1(\omega)/\ell_1}\int_0^{\xi_2(\omega)/\ell_2} dQ_1 dQ_2 \frac{1}{1+Q_1^2+Q_2^2},$$

where $\tilde{c}$ is an appropriate numerical factor of $O(1)$.

- MSD's are expressed (Diffusion by driving force):

$$\begin{aligned}
\ell_1^2\hbar^2 &= \lim_{s\to\infty}\langle (q(s)-q(0))^2\rangle = \lim_{T\to\infty}\sum_{s\le T} D_{1s} = \hbar^2 D_1^{(0)}\ell_1, \\
\ell_2^2\hbar^2 &= \lim_{s\to\infty}\langle (J(s)-J(0))^2\rangle = \lim_{T\to\infty}\sum_{s\le T} D_{2s} = \hbar^2 D_2^{(0)}\ell_1,
\end{aligned}$$

Self-Consistent Mean-Field Theory 3

- the factor $\frac{\xi_\mu(\omega)^2}{\ell_\mu^2} t_\mu$ is independent of μ:

$$\frac{\xi_1(\omega)}{\ell_1} = \frac{\xi_2(\omega)}{\ell_2}$$

- $t_\mu = t_1$, for $\mu = 1$:

$$\frac{\xi_1(\omega)^2}{\ell_1^2}(-i\omega\ell_1) = 1 - \frac{\tilde{c}}{\ell_2} \log\left[1 + \frac{\xi_1(\omega)^2}{\ell_1^2}\right].$$

$$\boxed{\xi_1(\omega = 0) \sim \ell_1 \mathrm{e}^{\ell_2/2\tilde{c}}}$$

$\Rightarrow$ 2DDS $\xi_{2dds} \sim \ell_{mfp} \mathrm{e}^{\pi \ell_{mfp}/2}$
(mean free path ℓ_{mfp} of 1DDS)

Self-Consistent Mean-Field Theory 4

Time-dependent diffusion constants D_{1s} and D_{2s}

- SM:

$$D_{1s} = K^2[\langle \sin^2 q_s\rangle + \mathrm{Re}\sum_{s'<s}\langle \sin q_{s'} \sin q_s\rangle]$$

$$D_{2s} = K^2\epsilon^2[\langle \cos^2 q_s\rangle \sin^2\phi_s + \mathrm{Re}\sum_{s'<s}\langle \cos q_{s'} \cos q_s\rangle \sin\phi_{s'} \sin\phi_s].$$

$$D_1^{(0)} = D_{cl}/\hbar^2 = K^2/2\hbar^2, \quad D_2^{(0)} = \epsilon^2 D_1^{(0)}/2$$

$$\ell_1 = D_1^{(0)} = \frac{D_{cl}}{\hbar^2}, \quad \ell_2 = \frac{\epsilon}{\sqrt{2}} D_1^{(0)}.$$

$$\boxed{p_\xi \propto \left(\frac{K}{\hbar}\right)^2 \exp\left[\mathrm{const.}\epsilon\left(\frac{K}{\hbar}\right)^2\right]}$$

Self-Consistent Mean-Field Thoery 5

- AM:

$$\ell_2^2\hbar^2 \sim W^2\epsilon^2\sum_{s,s'}\langle v(q_s)v(q_{s'})\rangle\langle\sin\phi_s\sin\phi_{s'}\rangle \sim W^2\epsilon^2\ell_1/2$$

$$\ell_1 \simeq \begin{cases} 1/W^2 & (W < W^*) \\ 1/W^{*2} & (W > W^*). \end{cases}$$

$$\ell_2 \sim \epsilon\sqrt{W^2\ell_1/2} \simeq \begin{cases} \epsilon/\sqrt{2} & (W < W^*) \\ \epsilon W/\sqrt{2} & (W > W^*). \end{cases}$$

$$\boxed{\xi(\epsilon, W) \simeq \begin{cases} c_0 W^{-2} exp\{c_1\epsilon\} & (W < W^*) \\ \xi_0^* exp\{c_2 W\epsilon\} & (W > W^*), \end{cases}}$$

Localization-delocalization transition 1($M \geq 2$)

$$
\begin{aligned}
H_{tot}(\hat{p},\hat{q},t) &= 2\cos(\hat{p}/\hbar) + Wv(\hat{q})\left[1+f(t)\right]\delta_t, \\
v(\hat{q}) &= \sum_{n\in\mathbb{Z}} \delta(q-n) v_n |n\rangle\langle n| \\
f(t) &= \frac{\epsilon}{\sqrt{M}} \sum_{j}^{M} \cos\omega_j t\delta_t, \overline{f(t)^2} = \epsilon^2/2
\end{aligned}
$$

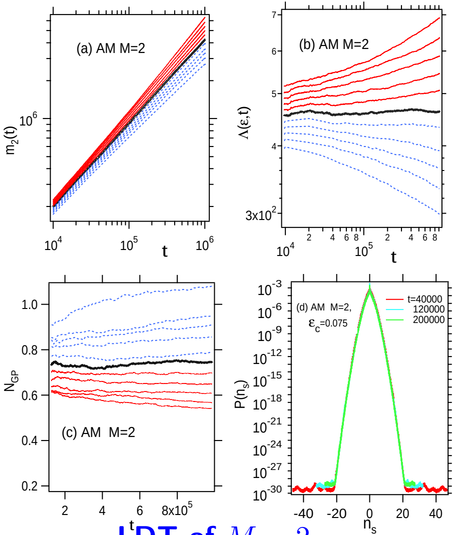
- Localization-delocalization transition (LDT) for $M \geq 2$?
  - Critical perurbation strength ϵ_c
  - Critical disorder strength W_c
  - subdiffusion exponent $\alpha(m_2 \sim t^{\alpha})$ at ϵ_c
  - critical exponent $\nu(\xi \sim |\epsilon-\epsilon_c|^{-\nu})$ at ϵ_c

Localization-delocalization transition 2($M \geq 2$)

- MSD and the scaled MSD

$$
\begin{aligned}
m_2 &\sim t^{\alpha} \\
\Lambda(\epsilon, t) &\equiv \frac{m_2(\epsilon, t)}{t^{\alpha}} \\
&= F((\epsilon_c - \epsilon)t^{\alpha/2\nu})
\end{aligned}
$$

The asymptotic function form: $F(x) \to |x|^{-2\nu}$.



LDT of $M=2$

Localization-delocalization transition 3($M \geq 2$)

- the non-Gaussian parameter (NGP) N_{GP}

$$N_{GP}(t) = \frac{1}{3}\frac{m_4(t)}{m_2(t)^2} - 1,$$

where $m_4(t) = \sum (n- < n >)^4 P(n,t)$.
$P_s(n_s(t),t) = P(n,t)dn/dn_s(t)$ at several t's as a function of scaled coordinate $n_s(t)$ by the spread of the wavepacket:

$$n_s(t) = \frac{n}{\sqrt{m_2(t)}} \propto \frac{n}{t^{\alpha/2}}$$

Evidently, Stretched Gaussian distribution:

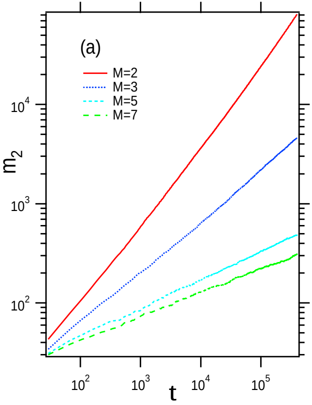
$$P(n_s) \sim \exp(-|n_s(t)|^{\beta}),$$

except for the range close to the origin of the critical state,

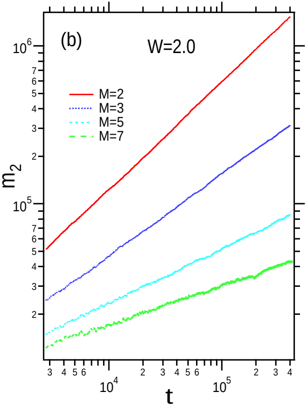
Localization-delocalization transition 4 (exponents:α)

- $M-$dependence of subdiffusion at the critical value ϵ_c

$$m_2(t) \sim t^{\alpha} \quad \epsilon = \epsilon_c,$$



SM



AM

One-parameter scaling theory and diffusion exponent α:

- In the long-time limit ($t \to \infty$),

$$m_2(t) \sim \begin{cases} \xi^2 & (\epsilon < \epsilon_c) \\ Dt & (\epsilon > \epsilon_c), \end{cases}$$

In the vicinity of LDT $\epsilon \simeq \epsilon_c$, we assume

$$\begin{cases} D \sim (\epsilon - \epsilon_c)^s & (\epsilon > \epsilon_c) \\ \xi \sim (\epsilon_c - \epsilon)^{-\nu} & (\epsilon < \epsilon_c,). \end{cases}$$

The exponents satisfy Wegner relation

$$s = (d-2)\nu$$

where d is spatial dimension.

- Scaling hypothesis:

$$m_2(t) = a^2 F_1(L_t/a, \xi/a),$$

with two-variable scaling function $F_1(x_1, x_2)$.

Localization-delocalization transition 6 (exponent:α)

Here a unique characteristic length L_t introduced by dynamics is given

$$L_t \sim t^{\sigma},$$

where σ is a dynamical exponent. If we set $a = \xi \ m_2$ scales like

$$\begin{aligned} m_2(t) &= \xi^2 F_1(t^{\sigma}/\xi, 1), \\ &= t^{2\sigma} F_2(t^{\sigma/\nu}(\epsilon - \epsilon_c)), \end{aligned}$$

where $F_2(x)$ is a one-variable scaling function.

$$2\sigma + \frac{\sigma s}{\nu} = 1,$$

Using Wegner relation it follows

$$\sigma = \frac{1}{d}.$$

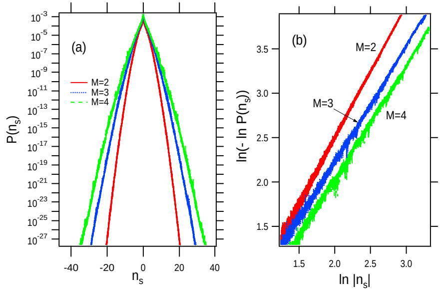
Therefore, at the critical point $\epsilon = \epsilon_c$ of LDT, MSD behaves subdiffusion

$$m_2(t) \sim t^{\alpha} = t^{\frac{2}{d}}$$

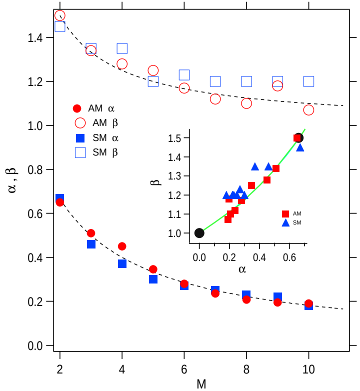
Localization-delocalization transition 7 (exponents:α, β)

- Scaling theory at the critical point ϵ_c for M colored case

$$\alpha_M = \frac{2}{d} = \frac{2}{M+1}, \beta_M = \frac{2}{(2-\alpha)} = \frac{M+1}{M}, 0 \le \alpha \le 1, 1 \le \beta \le 2$$



distribution form



(α, β) **relation**

$\alpha - \beta$ relation 1

•Master Eq. of probability distribution $P(n,t)$:

$$\frac{\partial}{\partial t}P(n,t) = \int_0^t dt^{'} \sum_{n^{'}} W(n, n^{'}, t - t^{'}) \Delta P(n^{'}, n, t^{'}),$$

where, $\Delta P(n^{'}, n, t^{'}) \equiv P(n^{'}, t^{'}) - P(n, t^{'})$. After caouse-graning,

$$\frac{\partial}{\partial t}P(x,t) = \int_0^t dt^{'} I(t - t^{'}) \frac{\partial^2}{\partial x^2} P(x, t^{'}),$$

$$I(t - t^{'}) = (1/2) \sum_{\Delta x} g(\Delta x, t)(\Delta x)^2$$

Laplace transform:

$$s\tilde{P}(x,s) = \tilde{I}(s) \frac{\partial^2 \tilde{P}(x,s)}{\partial x^2}$$

$\alpha - \beta$ relation 2

•Solution

$$\tilde{P}(x,s) = B(s)e^{-|x|\sqrt{s/\tilde{I}(s)}},$$

$$P(x,t) = \int_{\delta-i\infty}^{\delta+i\infty} dsB(s)e^{-|x|\sqrt{s/\tilde{I}(s)}+st},$$

In the $s \to 0$,

$$\sqrt{s/\tilde{I}(s)} \equiv f_0 s^{\alpha}$$

Saddle point of the equation $(0 < \alpha < 1)$:

$$\alpha|x|f_0 s^{\alpha} - st = 0,$$

In the limit $\alpha/t << 1$, $s \to 0$,

$$s^* = (\alpha f_0|x|/t)^{1/(1-\alpha)},$$

$$P(x,t) = Bexp\{-|x/x_s(t)|^{\beta}\}$$

$\alpha - \beta$ relation 3

For $t^\alpha < |x| < t$,

$$x_s(t) = \frac{t^\alpha}{f_0 \alpha^\alpha (1-\alpha)^{(1-\alpha)}}, \\ \sim t^\alpha,$$

Note: $P(x,t)$ satisfies the fractional diffusion Eq.

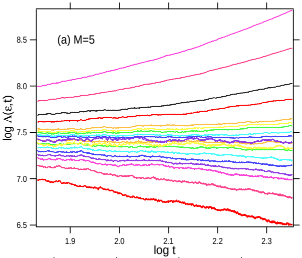
$$\frac{\partial^\alpha P(x,t)}{\partial t^\alpha} = \frac{1}{f_0^2} \frac{\partial^2 P(x,t)}{\partial x^2}$$

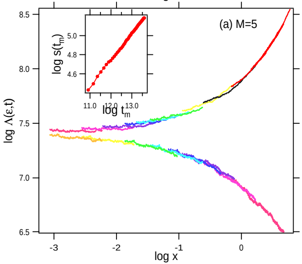
Note: In CTRW, for $a, b \in \mathbb{R}_+$,

$$\frac{\partial^a P(x,t)}{\partial t^a} = D_{a,b} \frac{\partial^b P(x,t)}{\partial x^b}$$

$$< x^2(t) > \sim t^{2a/b},$$

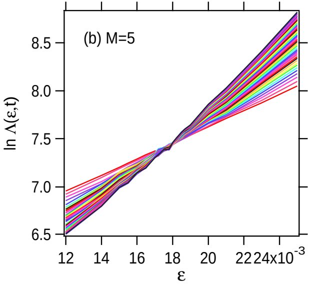
Localization-delocalization transition 8(AM, $M = 5$)





Scaled MSD around LDT of $M = 5$

- $W = 0.5 < W^*$



$\epsilon_c = 0.0175,\ \alpha \simeq 0.34$

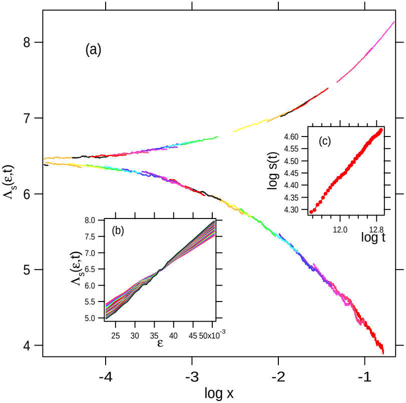
Inset:

$$
\begin{aligned}
s(t) &= \ln(\Lambda(\epsilon, t)/\Lambda_c(t))/(\epsilon_c - \epsilon) \\
&\propto t^{\alpha/2\nu}
\end{aligned}
$$

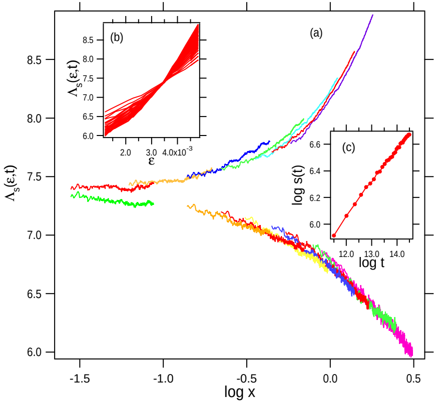
$\rightarrow \nu \simeq 0.62$

Localization-delocalization transition 9(AM:Scaling)

- $M = 3$, $W = 0.5$,

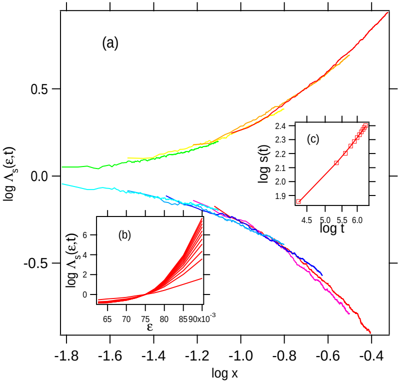


- $M = 7$, $W = 2.0$

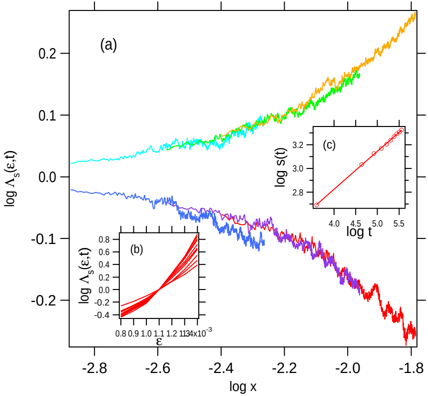


Localization-delocalization transition 10(SM:Scaling)

- $M = 3,\ K = 3.1,\ \hbar =$

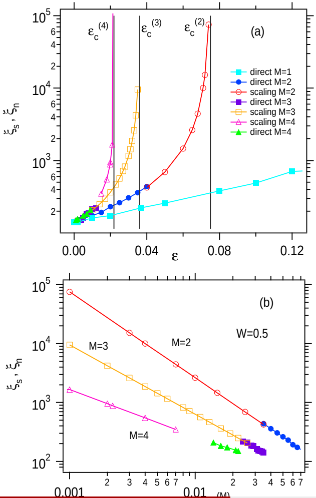


- $M = 7,\ K = 3.1,\ \hbar =$



Localization-delocalization transition 11 (localization length)

•AM with $W = 0.5$



$$\xi_n^{(M)}(\epsilon) = \sqrt{m_2(t \to \infty)}$$
$$\xi_s^{(M)}(\epsilon) = A_0^{(M)} |\epsilon - \epsilon_c^{(M)}|^{-\nu^{(M)}},$$

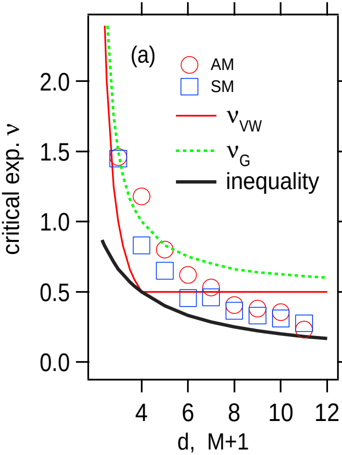
$$\xi_L^{(2)}(\epsilon) < \xi_L^{(3)}(\epsilon) < \xi_L^{(4)}(\epsilon) ...$$

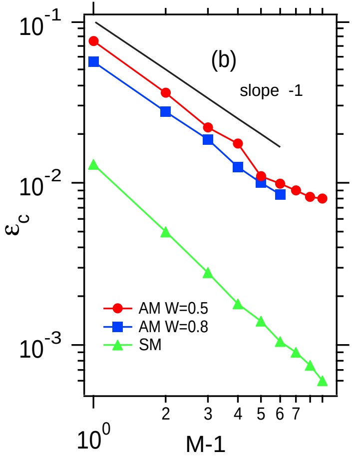
$$\epsilon_c^{(2)} > \epsilon_c^{(3)} > \epsilon_c^{(4)} > ....,$$
$$\nu^{(2)} > \nu^{(3)} > \nu^{(4)} > ....,$$

Localization-delocalization transition 12 (exponent ν)

- Critical exponent ν and critical value ϵ_c

$$\epsilon_c \propto (M-1)^{-\delta},$$





Localization-delocalization transition 13 (exponent ν)

Critical exponent ν of localization length:

•Self-consistent mean-field theory(Vollhardt and Wolfle 1980)

$$\nu_{VW} = \begin{cases} \frac{1}{d-2} & (2 < d < 4) \\ \frac{1}{2} & (d \geq 4) \end{cases}$$

• Semiclassical theory(Garcia etal 2008)

$$\nu_G = \frac{1}{2} + \frac{1}{d-2}$$

• Harris's ininequality(Harris 1974, Chayes etal 1986)

$$\nu > \frac{2}{d}$$

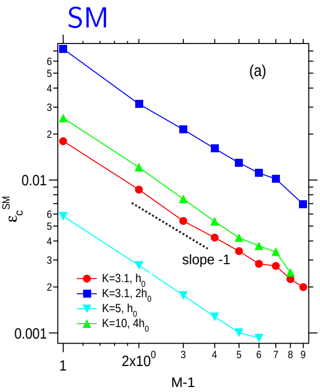
Localization-delocalization transition 14 (critical exponent)

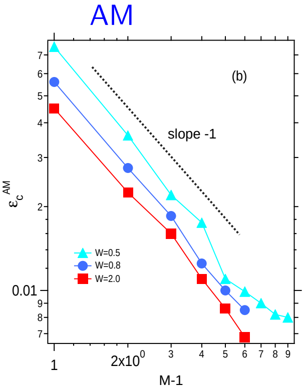
- d-dimensional random system

$$\xi \simeq |\epsilon - \epsilon_c|^{-\nu}, \quad \nu = 1/2 \;\; \text{in } d \to \infty(\text{MFT})$$

	M=2	M=3	M=4	M=5	M=6	M=7
SM($K = 3.1, \hbar = 0.24$)	1.37	0.95	0.70	0.50	0.50	0.40
Cchabe etal 08	1.58	1.15	–	–	–	–
Borgonovi etal 97	1.537	1.017	–	–	–	–
AM(W=0.5)	1.46	1.18	0.80	0.62	0.53	0.41
AM(W=2.0)	1.48	1.01	0.88	0.65	0.57	0.49
	d=3	d=4	d=5	d=6	d=7	d=8
Markos 06	1.57	1.12	0.93	–	–	–
Garcia etal 07	1.52	1.03	0.84	0.78	–	–
Slevin etal 14	1.57	1.15	0.97	–	–	–
Tarquini 17	1.57	1.11	0.96	0.84	–	–

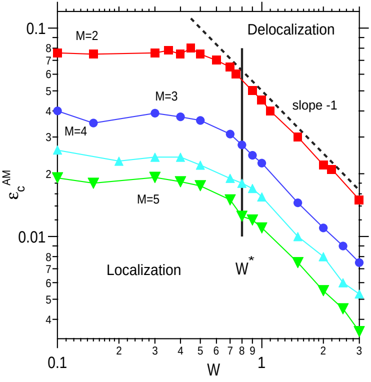
- M-dependence of ϵ_c



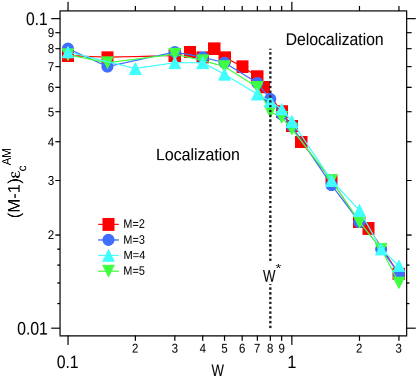


$$\epsilon_c \sim \frac{1}{M-1} \quad (M \geq 2)$$

- Critical curves



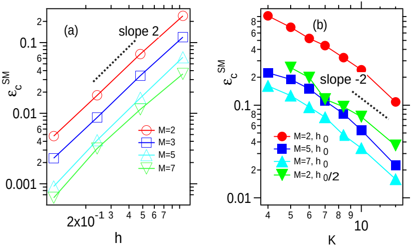
- Scaled Critical curves



$$\epsilon_c \Rightarrow (M-1)\epsilon_c$$

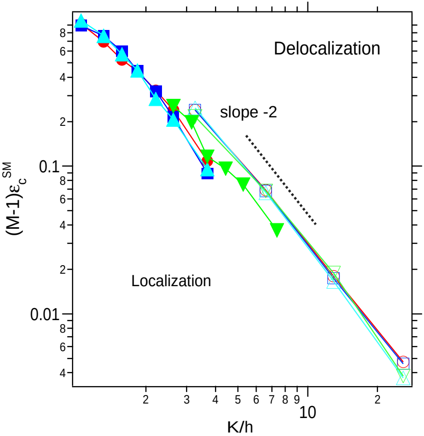
Localization-delocalization transition 17 (SM: ϵ_c)

- $\hbar/K-$dependence



$$\epsilon_c \Rightarrow (M-1)\epsilon_c$$

- Scaled diagram



Conclusions

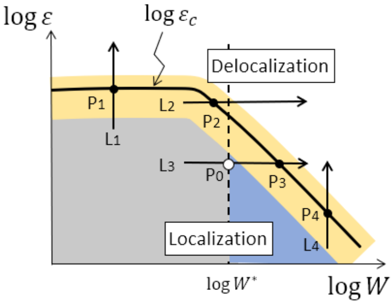
- $M = 1$:monochromatically perturbed case
  - The localization length can be expressed by the localization length ξ_0 of the unperturbed system
  - It can be approximately explained by SCT for anisotropic 2DDS
  - It corresponds well with the localized phenomena of the SM
  - Localized phenomena are different at W^*
  - W^* can be explained in the Maryland transform.
- $M \geq 2$:polychramatically perturbed cases
  - Critical strength ϵ_c, LDT observed
  - Diffusion exponent α and β at $\epsilon = \epsilon_c$ is close to that of OPST.
  - $M-$dependence of ν deviates from that of the self-consistent MFT for $M \to \infty$
  - $M-$dependence of ϵ_c does not depend on W

Remaining problems

- $M = 1$:monochromatically perturbed case
  - Relationship with the perturbed Anderson model (time-continuous system)
  - Scaling properties of localization processes
- $M \geq 2$:polychramatically perturbed cases
  - Derivation of the $M-$dependence of ϵ_c:Limit of application of SCT
  - Relationship with polychromatically perturbed Anderson model (time-continuous system): Does the LDT occur in tha case $M \geq 2$?
  - Does the LDT occur in monochromaticaaly perturbed 2DDS?

Localization-delocalization transition (AM: phase diagram)

- Critical curve



$$\epsilon_c \Rightarrow (M-1)\epsilon_c$$

- $M \geq 2,\ W < W^*(L_3)$

$\epsilon < \epsilon_c$	Ballistic $\to$ Localization
$\epsilon = \epsilon_c$	Ballistic $\to$ Subdiffusion
$\epsilon > \epsilon_c$	Ballistic $\to$ Diffusion

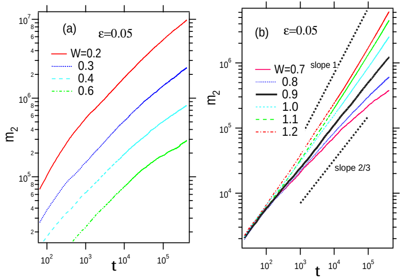
- $M \geq 2, W > W^*(L_4)$

$\epsilon < \epsilon_c$	Diffusive $\to$ Localization
$\epsilon = \epsilon_c$	Diffusive $\to$ Subdiffusion
$\epsilon > \epsilon_c$	Diffusive $\to$ Diffusion

Localization-delocalization transition (AM: W_c)

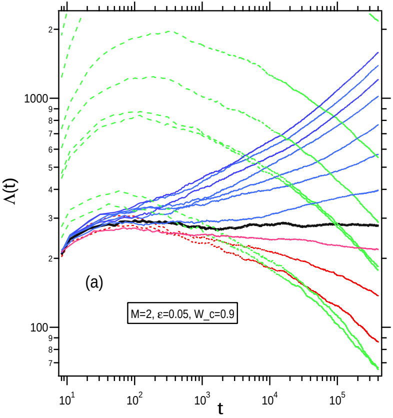
- AM, $M = 2$, $\epsilon = 0.05$

MSD



$m_2 \propto t^{2/3}$ at $W = W_c (= 0.9)$

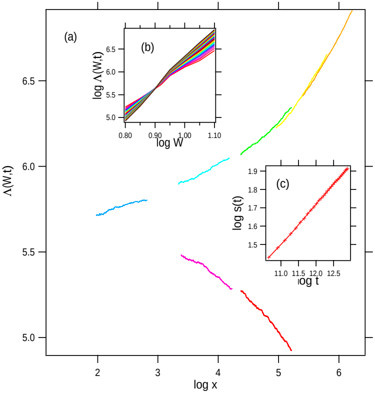
Scaled MSD



green $\rightarrow$ red $\rightarrow$ blue with increasing W

Localization-delocalization transition (AM: Scaling)

OPST:AM, $M = 2, \epsilon = 0.65$



$$\log \Lambda(W,t) = F(x), x = (W_c - W)t^{\alpha/2\nu}.$$

- $F(x)$:differentiable scaling function
- α:diffusion index
- Around $W \simeq W_c$:

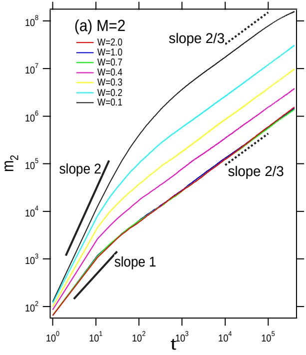
$$F(x) = F(0) + s(t)(W_c - W) + \ldots\ldots$$

$$s(t) = \frac{\log \Lambda(W,t) - \log \Lambda_c}{|W_c - W|} \propto t^{\alpha/2\nu}$$

Localization-delocalization transition (AM: ϵ_c)

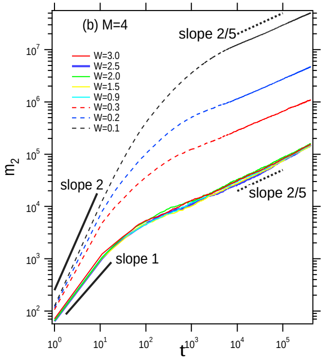
- $W-$dependence of subdiffusion at ϵ_c for different W

$M = 2$



$$m_2 \propto t^{2/3}$$

$M = 4$



$$m_2 \propto t^{2/5}$$

References-1

Perterbed Anderson map and Standard map

•[HY04] H. Yamada and K. S. Ikeda, Phys. Lett. A **328**, 170 (2004).
•[HY10] H.S.Yamada and K.S.Ikeda, Phys. Rev. E **82**, 060102(R)(2010).
•[HY15] H.S. Yamada, F. Matsui and K.S. Ikeda, Phys. Rev. E **92**, 062908(2015). H. S. Yamada and K. S. Ikeda, in preparation
•[HY17] H.S.Yamada and K.S.Ikeda, Phys. Rev. E **97**, 012210(2018).

Perturbed Anderson model

•[HY98] H. Yamada and K.S. Ikeda, , Phys. Lett. A 248, 179-184 (1998).
•[HY99] H.Yamada and K.S. Ikeda, Phys. Rev. E 59, 5214-5230 (1999).
•[HY03] H. Yamada, Proc. of Japan-Italy Joint Workshop on Quantum Open System, Quantum Chaos and Quantum Measurement: Fundamental Aspects of Quantum Physics, Ed. by L. Accardi and S. Tasaki, 80-99 (World Scientific 2003)

References-2

Perterbed standard map

•[casati79] G. Casati, B. V. Chirikov,F. M. Izraelev, J.Ford, *Stochastic behavior of a quantum pendulum under a periodic perturbation* (Springer-Verlag,Berlin,1979) ed. by G.Casati and J.Ford .pp334.

•[casati89] G.Casati, I.Guarneri and D.L.Shepelyansky, Phys. Rev. Lett. **62**, 345(1989).

•[borgonovi97] F.Borgonovi and D.L.Shepelyansky, Physica D**109**, 24 (1997).

•[chabe08] J.Chabe, G.Lemarie, B.Gremaud, D.Delande, and P.Szriftgiser, Phys. Rev. Lett. **101**, 255702(2008).

•[wang09] J. Wang and A. M. Garcia-Garcia, Phys. Rev. E **79**, 036206(2009).

•[lemarie10] G. Lemarie, H.Lignier, D.Delande, P.Szriftgiser, and J.-C.Garreau, Phys. Rev. Lett. **105**, 090601(2010).

References-3

Perterbed standard map

•[tian11] C. Tian, A. Altland, and M. Garst, Phys. Rev. Lett. **107**, 074101(2011).

•[manai15] I. Manai, J-F. Clement, R. Chicireanu, C. Hainaut, J. C. Garreau, P. Szriftgiser, and D. Delande, Phys. Rev. Lett. **115**, 240603(2015).