Delocalization by Correlated disorder in One-Dimensional Disordered Systems

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Contents:

Anderson localization in 1DDS

- Anderson model (tight-binding model),

$$-t\phi_{n+1} - t\phi_{n-1} + V_n\phi_n = E\phi_n$$

  - V_n: spatial disorder (diagonal disorder)
  - t: hopping integral ($t = 1$)
  - E: energy
  - ϕ_n: amplitude at site $n \in \mathbb{Z}$

- A typical $\delta-$correlation with disorder strength W
  $V_n = Wv_n$, v_n i.i.d. $v_n \in [-1, 1]$
- Anderson localization
  - p.p.spectrum (level clustering)
  - exponential decay of wavefunction

$$\phi_n \sim \exp\left(-\frac{n}{2\xi_{1d}}\right), \text{LL: } \xi_{1d} \approx \frac{4 - E^2}{< V_n^2 >} \text{ for } W << 1$$

Other Representation

• Floquet picture (Transfer matrix)

$$\begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix} = T_n \begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix}, T_n = \begin{pmatrix} \frac{E-V_n}{t} & -1 \\ 1 & 0 \end{pmatrix}$$

where $\phi_n = \langle n|\Phi\rangle$ is the amplitude of the wavefunction $|\Phi\rangle = \sum_n \phi_n C_n^\dagger |0\rangle$ at the site n.

• Herbert-Jones-Thouless formula:

$$\begin{aligned} L(E) &= \lim_{N\to\infty} \frac{1}{N} \ln \left| \frac{\phi_N(E)}{\phi_0} \right| \\ &\simeq \int_{-\infty}^{\infty} dE^{'} \rho(E^{'}) \log |E^{'} - E| + i\pi \int_{-\infty}^{\infty} dE^{'} \rho(E^{'}) \theta(E^{'} - E), \\ &= \gamma(E) + i\pi I(E) \end{aligned}$$

where $\gamma(E)$, $\rho(E)$ and $I(E)$ are the Lyapunov exponent, the averaged DOS and the integrated DOS.

Participation Ratio

- Diagonalization of Hamitonian matrix

$$H|\phi_m\rangle = E_m|\phi_m\rangle, H = \begin{pmatrix} V_0 & -t & \cdots & \cdots & -te^{i\theta} \\ -t & V_1 & & & \vdots \\ \vdots & & V_n & & \vdots \\ \vdots & & & \ddots & -t \\ -te^{-i\theta} & \cdots & \cdots & -t & V_N \end{pmatrix}$$

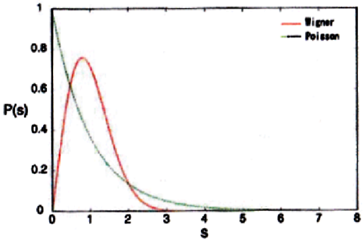
- Participation ratio (PR)$P(E_m)$, inverse participation ratio (IPR):

$$P(E_m) = \left[\sum_n^N |\phi_m(n)|^4\right]^{-1} \propto N^d, \quad d = \begin{cases} 0 & \text{localization} \\ D & \text{critical} \\ d & \text{extended} \end{cases}$$

Distribution of Nearest-neighbour spacing

• Nearest-neighbour energy eigenvalues $(s = |E_{n+1} - E_n|/\Delta)$

$$P(s) = \begin{cases} e^{-s} & \text{localization} \\ \frac{\pi}{2} s e^{-\pi s^2/4} & \text{GOE : level repulsion} \end{cases}$$



Poisson and Wigner distribution

Mathematical Discription

An ergodic operator on $\ell^2(Z)$:

$$(H_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n)$$

Anderson Localization: p.p. spectrum with exponentially decaying eigenfunctions characterized by positive difinite of $\gamma(E) > 0$ for a.a. ω.
Lyaponov exponent $\gamma(E)$:

$$\gamma(E) \equiv \lim_{n\to\infty} \frac{\ln(|u(n+1)|^2 + |u(n)|^2)}{2n}$$

The closure of the set of positivity of the Lyapunov exponent equals the essential support of the a.c. part of the spectral measure of the operator.
Kotani theory: $\gamma(E) > 0$ (and thus there is no a.c. spectrum) unless a ergodic potential V_ω is deterministic in his sense. $u(0) = u(1) = 1$

Mathematical Discription 2

•One-electron problem under a random potential V in 1d

$$i\frac{\partial\phi}{\partial t} = -\Delta + \lambda V\phi$$

•absence of absolute continuous spectrum (Ishii 1973)

$$\Sigma^{\omega}_{ac}(dE) = \emptyset$$

•pure point spectrum and exponential localization for any λ (I.Goldseid,S.Molchanov,L.Pastur, 1977)

$$\Sigma^{\omega}(dE) = \Sigma^{\omega}_{pp}(dE)$$

• $d-$dim, Absence of diffusion for large disorder or low energy (J.Frolich, T.Spencer 1983)

$\lambda >> 1 \rightarrow \text{localization}, \lambda \neq 0 \rightarrow \text{localization at the edge of the spectrum}$

Correlated Disorder

•short-range correlation $(n_{cr} < \infty, \alpha = 0)$

$$\begin{aligned} C(n) &= < V_{n+n_0} V_{n_0} > \sim W^2 \exp\left(-n/n_{cr}\right) \\ S(f) &= \frac{1}{N} \left| \sum_{n=0}^{N} V_n e^{-i2\pi f n/N} \right|^2 \sim 1/f^{\alpha} \end{aligned}$$

$\Rightarrow$ same as $\delta-$correlated case

• long-range correlation $(n_{cr} \to \infty,\ 0 < \beta < \infty,\ 0 < \alpha < 1)$

$$\begin{aligned} C(n) &\sim W^2 \frac{1}{n^{\beta}}, \\ S(f) &\sim 1/f^{\alpha} \end{aligned}$$

$\Rightarrow$ same as $\delta-$correlated case (exponential localization)

Correlated Disorder-an example-

•1DDS with a Gaussian random potential $V(x)$ (Iomin, PRE 09):

$$-\frac{d^2\phi(x)}{dx^2} - V(x)\phi(x) = E\phi(x),$$

$$C_\kappa(x_2 - x_1) = \langle V(x_1)V(x_2)\rangle = \frac{C_\kappa}{|x_2 - x_1|^{1+\kappa}},$$

where $C_\kappa = 2\sigma^2/\Gamma(-\kappa)$, for $-1 < \kappa < 0$. $\sigma^2 =< V(0)^2 >$.

• Rate of the exponential growth:

$$\boxed{\gamma_{gm}(E) \equiv \lim_{x\to\infty} \frac{\ln\left\langle \phi^2(x)\right\rangle}{x},} \Rightarrow \gamma_{gm}(0) \sim (2\sqrt{\sigma})^{4/(3-\kappa)} > 0.$$

under appropriate boundary conditions at $x = 0$, $\phi(x = 0)$ and $\phi^{'}(x = 0)$.

•Note that:

$$\gamma_{gm}(E) \geq \gamma_{am}(E) \equiv \lim_{x\to\infty} \frac{\left\langle \ln \phi^2(x)\right\rangle}{x}$$

Average

•A note on average operations : Definition by generalized qth moment moment-definition (MD) γ_k, log-definition (LD) β_k

$$
\begin{aligned}
\gamma_k &\equiv \lim_{n\to\infty} \frac{\log < |\psi_n|^k >}{kn}, \\
\beta_k &\equiv \lim_{n\to\infty} \frac{< \log |\psi_n|^k >}{kn}
\end{aligned}
$$

$$
0 \le \beta_k \le \gamma_k,
$$

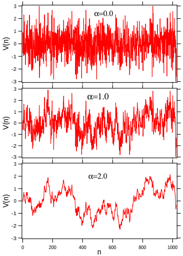
Self-averaging enabled and well-defined Lyapunov exponent β_2 can be an order parameter of the Localization problem. Even if $\gamma_2 > 0$, $\beta_2 = 0$ and the localization length can be infinite. Conversely, even if $\beta_2 = 0$, it is likely that $\gamma_2 > 0$ may be obtained, and it is not necessarily extended.

Long-range correlated disorder (FFM)

- Long-range correlated disorder (Fourier filtering method '98)

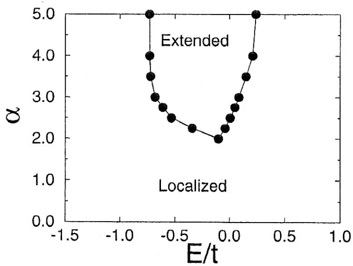
$$V(n) = C \sum_{k=1}^{N/2} \left[k^{-\alpha} \left(\frac{2\pi}{N} \right)^{1-\alpha} \right]^{1/2} \cos\left(\frac{2\pi k n}{N} + \varphi_k \right),$$

  - φ_k: i.i.d. $\varphi_n \in [0, 2\pi]$
  - α: a spectrum index ($\alpha \geq 0$) $\Rightarrow S(f) \sim 1/f^{\alpha}$



energy sequences

$(\alpha = 0, 2, 2.5)$

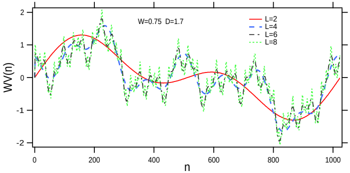


phase diagram

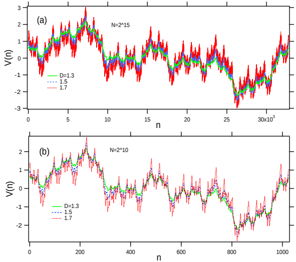
- Long-range correlated disorder (Weierstrasse function)

$$v_n = C \sum_{k=0}^{L} \frac{\sin(2\pi a^k n/N + \varphi_k)}{a^{(2-D)k}}, \quad V(n) = W v_n$$

  - φ_k: i.i.d. $\varphi_n \in [0, 2\pi]$
  - a: a constant value $(a > 1)$ related the scale-invariance
  - C: Normalization constant, $\langle v_n \rangle = 0$, $\langle (\Delta v_n)^2 \rangle = 1$
  - D is a fractal dimension $(1 < D < 2)$



potential sequences

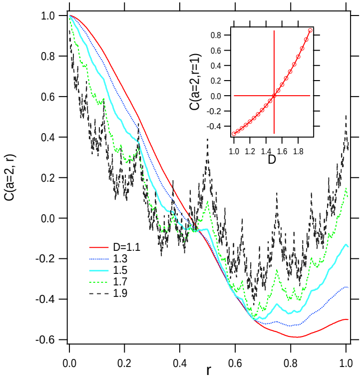


potential sequences

• Autocorrelation function: $a = 2$

$$C(a,r) = \langle v_{n_0} v_{n_0+r} \rangle$$

$$= \frac{\sum_{k=0}^{\infty} a^{-2(2-D)k} \cos(\pi a^k r)}{\sum_{k=0}^{\infty} a^{-2(2-D)k}},$$



negative correlation

Normalized localization length

•Lyapunov exponent:

$$\gamma_N(W,D) = \frac{ln(|\phi_{N+1}|^2 + |\phi_N|^2)}{2N}, \gamma_{N,am} = \langle \gamma_N \rangle$$

•Localization length (LL):

$$\xi_N = \frac{1}{\gamma_{N,am}} \leq \left\langle \frac{1}{\gamma_N} \right\rangle \equiv \frac{1}{\gamma_{N,hm}}$$

•Normalized localization length (NLL):

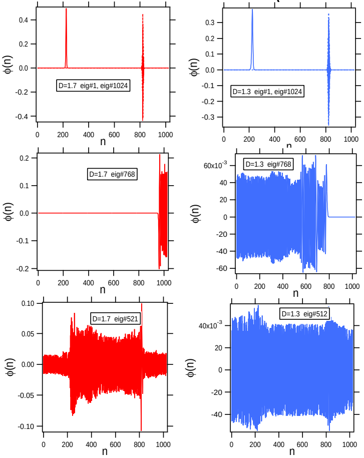
$$\Lambda_N \equiv \frac{1}{\langle \gamma(N) \rangle N} = \frac{\xi(N)}{N}.$$

•For $N >> 1$,

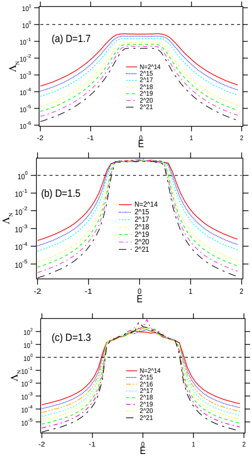
$$\Lambda_N \sim N^{\delta} \quad \begin{cases} \delta < 0 & \text{localization} \\ \delta = 0 & \text{critical} \\ \delta > 0 & \text{extended} \end{cases}$$

- Some eigenstates

$$\gamma_N = \left\langle \frac{ln(|\phi_{N+1}|^2 + |\phi_N|^2)}{2N} \right\rangle,$$

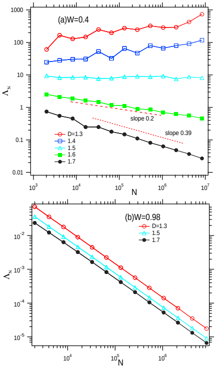


$a = 2$

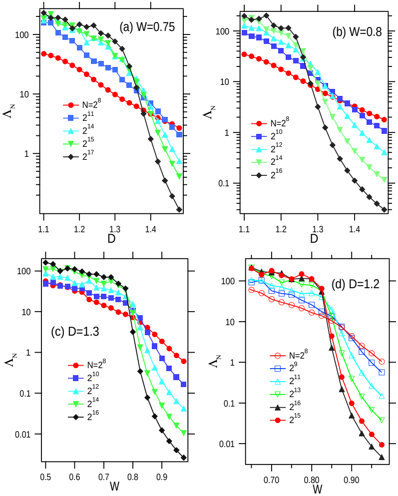


Long-rang correlated disorder (Weierstrasse function)-4

- Normalized localization length (NLL): $\Lambda_N \equiv \frac{1}{<\gamma(N)>N} = \frac{\xi(N)}{N}$.



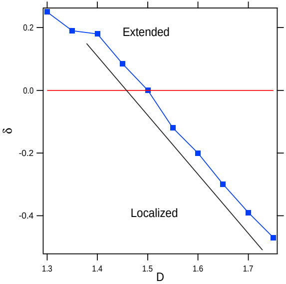
$N-$**dependence**



$D/W-$**dependence**

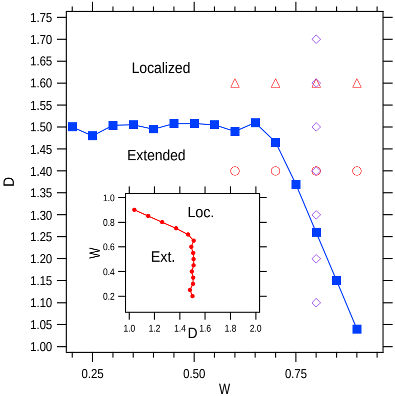
Long-range correlated disorder (Weierstrasse function)-5

• $\Lambda_N \sim N^{\delta}(W = 1.0)$



$D-$dependence of the δ

$D_c \simeq 3/2$ for LDT



Phase diagram

Correlated chaotic field-1

•Modified Bernoulli (MB) map

$$X_{n+1} = \begin{cases} X_n + 2^{B-1}(1-2b)X_n^B + b & (0 \le X_n < 1/2) \\ X_n - 2^{B-1}(1-2b)(1-X_n)^B - b & (1/2 \le X_n \le 1), \end{cases}$$

• the symbolized sequence

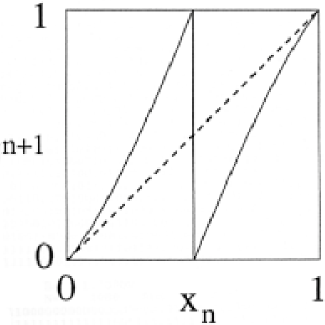
$$\begin{cases} 0 \le X_n < 1/2 \to & V_n = -W \\ 1/2 \le X_n < 1 \to & V_n = W. \end{cases}$$

• Long-range correlated disorder

$$S(f) \sim \begin{cases} f^0 & 1 \le B < 3/2 \\ f^{-\alpha}(f << 1) & 3/2 \le B \le 3, \end{cases} \alpha = \frac{2B-3}{B-1}$$

Correlated chaotic field-1a

•Modified Bernoulli map



$B = 1.7, \epsilon = 0$

- Renewal process N_t:

$$Var(N_t) = \langle (N_t - \langle N_t \rangle)^2 \rangle \sim t^\delta$$

- pausing-time distribution:

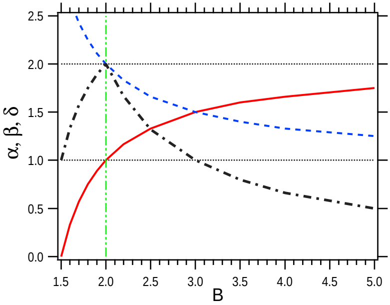
$$P(m) \sim m^{-\beta}, \beta = \frac{B}{B-1}$$

Correlated chaotic field-2

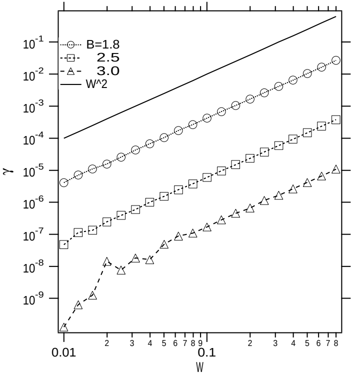
- Renewal process:

$Var(N_t) =< (N_t- < N_t >)^2 >\sim t^\delta$

$< \gamma_N >\to \gamma_N$ (abbreviation)



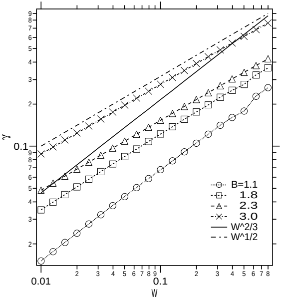
$B-$dependence of the exponents, α, δ



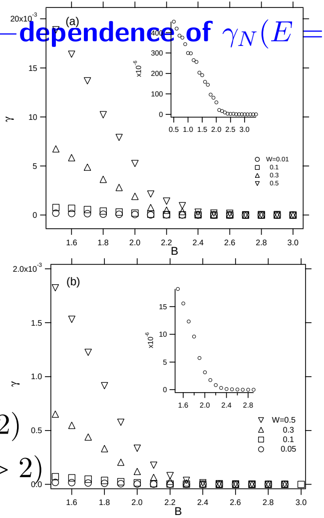
$W-$dependence of $\gamma_N(E=0)$

Correlated chaotic field-3

- $W-$**dependence of** $\gamma_N(E=2)$

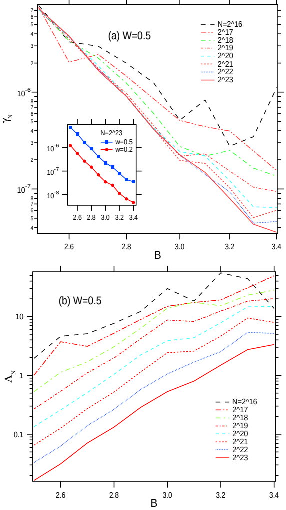


- $B-$**dependence of** $\gamma_N(E=0)$

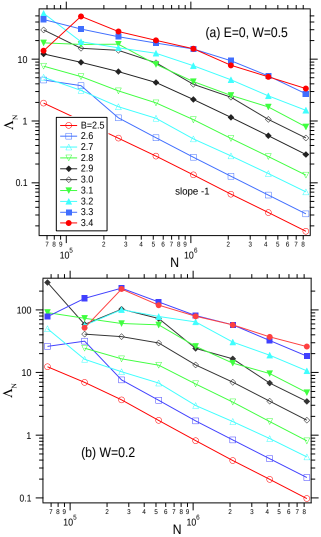


$$\gamma \propto \begin{cases} W^2 \;\; (E=0) & '81 \to W^2 (B>2) \\ W^{2/3} \; (E=2) & '84 \to W^{1/2} (B>2) \end{cases}$$

- Lyapunov exponent and NLL



$B-$**dependence**



$N-$**dependence**

• $B-$**dependence of** $\gamma_N(E=0)$

$$\gamma(E=0) \propto \begin{cases} W^2(a-bB) \ \ (B \leq 2, a > 2b) \\ W^2\mathrm{e}^{-cB} \ (B > 2) \to 0 (B \to \infty) \end{cases}$$

Furstenberg Theorem 1

Theorem 1

Furstenberg's theorem: Let G be a noncompact subgroup of $SL(R,2)$ such that no subgroup of G of finite index is reducible. (F-conditions) Then

$$\lim_{N\to\infty} \frac{1}{n} \log ||\Pi_i^n T_i u_0|| = 2\gamma > 0$$

w.p. 1 for all $u_0 \in (R^2 - \{0\})$.

Remark 0.1

GF-conditions (generalized F-condition): 1) G^i satisfies $F-$condition, 2)

$$\int_{SL(R,2)} |Q^i| d\mu^1(Q^i) < \infty.$$

Furstenberg Theorem 2

(W, m): m times iteration of W
Joint probability of the clusters $\{(W, m), (-W, \ell)\}$:

$$P(m, \ell) \sim m^{-\beta} \ell^{-\beta}, \beta = \frac{B}{B-1}$$

trabsfer matrix of the events

$$\tilde{T}_k \equiv T_+^{m_k} T_-^{\ell_k},$$

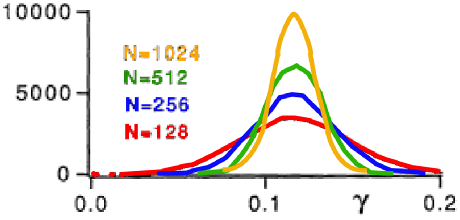
$$T_+ = \left[\begin{array}{cc} E - W & -1 \\ 1 & 0 \end{array} \right], \quad T_- = \left[\begin{array}{cc} E + W & -1 \\ 1 & 0 \end{array} \right],$$

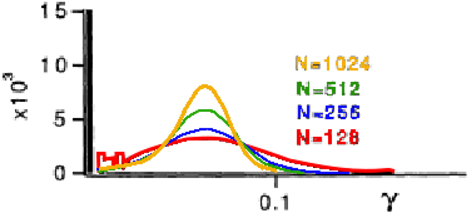
•Lyapunov exponent:

$$\begin{aligned} & \gamma_\infty(B) = \langle \gamma_\infty(B, \omega) \rangle_\omega \\ \simeq & \left\langle \lim_{M \to \infty} \frac{1}{2M < m >} \log ||\Pi_k^M \tilde{T}_k u_0|| \right\rangle_\omega \\ \Rightarrow & \gamma_\infty(B) = 0 (B \to \infty) \end{aligned}$$

- **Distribution of $\gamma_N(E=0)$**

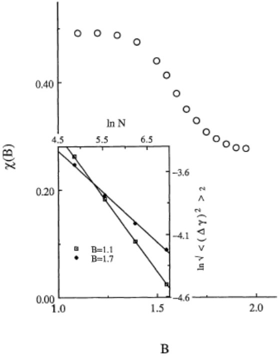
$B=1.1,\ B=1.7$





- **$N-$dependence of fluctuation:**

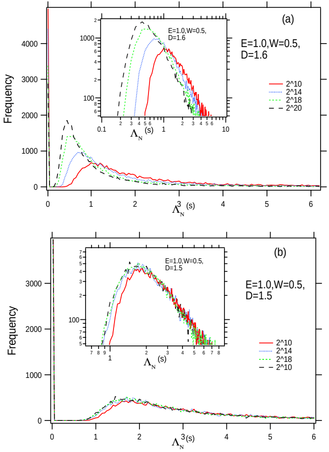
$\langle(\Delta\gamma)^2\rangle \sim N^{-\chi}$

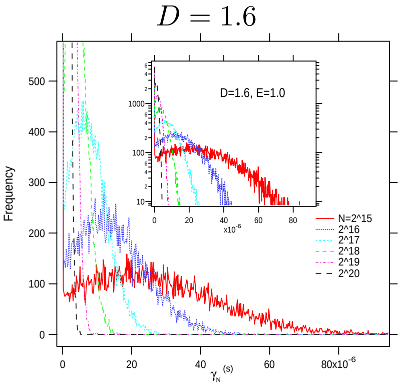


central limit theorem (CLT:$\chi=1/2$) $\to$ slow convergence

Convergence-2 (Weierstrass function)

- **Distribution of λ_N and NLL Λ_N**





long-tail

Conclusions

• **Long-range correlated disorder characterized spectrum index α**

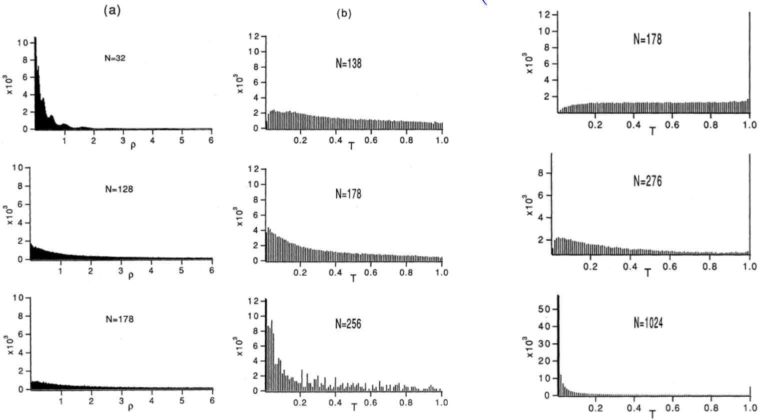
$$S(f) = \frac{1}{N}\left|\sum_{n=0}^{N} V_n e^{-i2\pi fn/N}\right|^2 \sim 1/f^{\alpha}$$

Conjectures$(\alpha = \frac{2B-3}{B-1}$, $\alpha = 5 - 2D)$

- $1/2 < \alpha < 1(5/3 < B < 2,\ 2 < D < 9/4)$
  - exponential localization, $\gamma_\infty > 0$
  - slow convergence(Convergence changes from CLT)
- $\alpha = 1(B = 2,\ D = 2)$
  - Exponential localization, no LDT
  - Parameter dependence (functional form) of γ_∞ changes
- $\alpha = 2(B = \infty, D = 3/2)$
  - Localiation-Delocalization transition (LDT) ??

Convergence-3(MB-system)

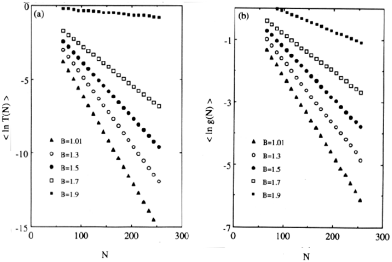
- **Distribution of $\rho_N(E=0)$ and $T(N)$ ($B=1.01.$ $B=1.7$)**



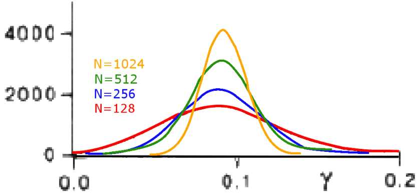
Anomalous fluctuation

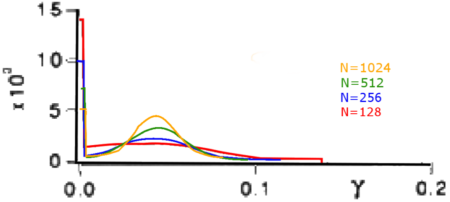
• **Distribution of L-exp. of $T(N)$**

• Transmition rate $T(N)$ and Thouless number $g(N)$



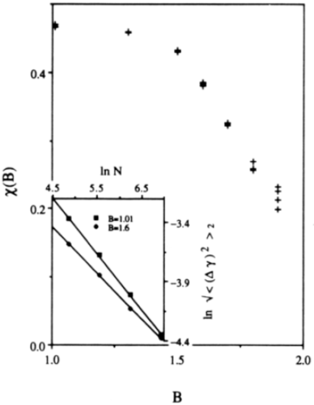
Exponential localization



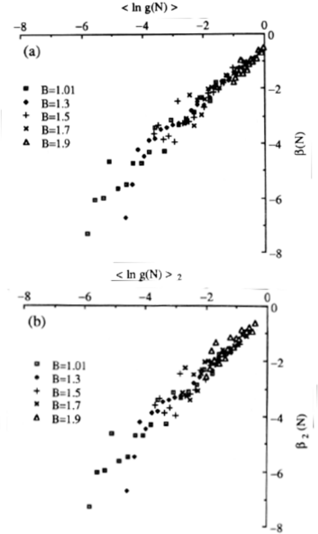


Convergence-5(MB-system)

- $\langle (\Delta\gamma)^2 \rangle \sim N^{-\chi}$



$B-$**dependence** $\Rightarrow$
Slow convergence



$\beta = \frac{\partial < \ln T >}{\partial lnN}$

Similar models-1

- Lloyd model (Lorentz distribution or Caucy distribution)

$$P_\delta(v_n) = \frac{1}{\pi}\frac{\delta}{v_n^2+\delta^2}, \langle v_n^2 \rangle = \infty$$

$$\gamma(E) = \cosh^{-1}\left[\frac{\sqrt{(2+E)^2+\delta^2}+\sqrt{(2-E)^2+\delta^2}}{4}\right]$$

Note that it is stable distribution

$$\int_{-\infty}^{\infty} P_t(x-y)P_s(y)dy = P_{t+s}(x)$$

- $\delta-$impurity model

$$V(x) = \sum_n v_n \delta(x-n), P(v_n)\ i.i.d$$

Similar models-2

•off-diagonal model

$$-t_{n+1}\phi_{n+1} - t_{n-1}\phi_{n-1} + V_n\phi_n = E\phi_n,$$

$$\begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix} = T_n \begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix}, T_n = \begin{pmatrix} \frac{E-V_n}{t_{n+1}} & -\frac{t_n}{t_{n+1}} \\ 1 & 0 \end{pmatrix}$$

• Chiral symmetry (particle-hole symmetry):invariant under transform $(E, \phi_n) \to (-E, (-)^n\phi_n)$, then for Ricatti variable $R_n = \frac{\phi_n}{\phi_{n-1}}$, $E = 0$

$$\phi_{n+1} = -\frac{t_{n-1}}{t_{n-1}}\phi_{n-1},$$

• stretched exponential growth

$$\log|\phi_n| \propto \sqrt{n}, \quad |\phi_n| \propto e^{C\sqrt{n}},$$

$\Rightarrow \gamma = 0$ Extended

Off-diagonal model-1

•Long-range off-diagonal disorder

$$t_{n,m} = \frac{V}{|n-m|^{\alpha}},$$

normalization

$$N^* \equiv d \int_1^{N^{1/d}} dr r^{d-1} r^{-\alpha} = \frac{N^{1-\alpha/d} - 1}{1 - \alpha/d}$$

Off-diagonal model-2

•In a limit $N \to \infty$,

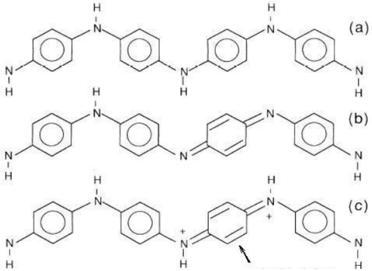
$$\begin{cases} \frac{\alpha}{d} > 1 & N^* \to \frac{1}{\alpha/d-1} < \infty(\text{extensive}) \\ \frac{\alpha}{d} = 1 & N^* \to \log N(\text{non} - \text{extensive}) \\ 0 \le \frac{\alpha}{d} < 1 & N^* \to \frac{N^{1-\alpha/d}}{1-\alpha/d}(\text{non} - \text{extensive}) \end{cases}$$

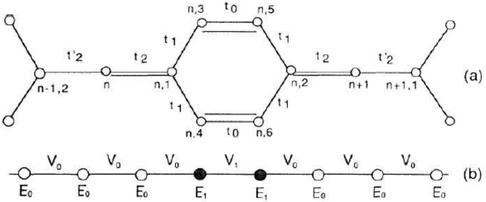
for $d = 1$

$$\begin{cases} \alpha < 1 & \text{subdiffusion} \\ \alpha = 1 & \text{normal diffusion} \\ 1 < \alpha < \alpha_c & \text{superdiffusion} \\ \alpha_c < \alpha & \text{ballistic motion} \end{cases}$$

Applications-1

- polymer (polyaniline)

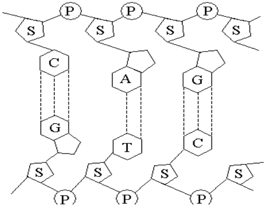


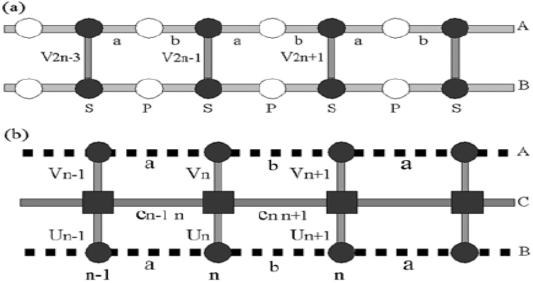


dimer model

Applications-2

- DNA





Simple ladder models of DNA

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Remaining problems

•acoustic models
•optical models
•electric field
$\Rightarrow$ Strak ladder localization
•Higher dimension, Many body interaction, Nonlinear effect, Conduction formula, ...

Dynamical Localization in static 1DDS

$$i\hbar\frac{\partial\phi(n,t)}{\partial t} = \phi(n+1,t) + \phi(n-1,t) + Wv(n)\phi(n,t),$$

$\phi(n, t = 0) = \delta_{n,N/2}$, $\hbar = 1$

•the mean square displacement (MSD),

$$m_2(t) = \sum_n (n - n_0)^2 \left\langle |\phi(n,t)|^2 \right\rangle \sim t^\sigma$$

diffusion exponent

$$\begin{cases} \sigma = 0 & \text{localization} \\ 0 < \sigma < 1 & \text{subdiffusion} \\ \sigma = 1 & \text{normaldiffusion} \\ 1 < \sigma < 2 & \text{superdiffusion} \\ \sigma = 2 & \text{ballistic motion} \\ \sigma = 2/3 & \text{MIT in 3DDS?} \end{cases}$$

- **Definition:**

$$||\phi_n(t)||_D \equiv \sqrt{m_2(t)}$$

- **Dynamical localization (DL):** if for any $\phi_n(0)$ with $\phi_n(0) \leq ce^{n\rho}$,

$$\overline{\lim_t} \sum_{n \in Z} |n|^s |\phi_n(t)|^2 < \infty,$$

DL $\rightarrow$ p.p. spectrum, but p.p. spectrum $\neq$ D.L.

Many cases of pure point spectrum:
•Anderson model (De Bievre-Germinet 1998, Damanik-Stollmann 2001...)
•Harper model (Germinet-Jitomirskaya 2001, Bourgain 2007):
$V(n) = \lambda \cos(2\pi\alpha n + \theta)$, α Diophantine, $\lambda > 2$, a.e. $\theta \in R$.
•Maryland model (Bellissard-Lima-Scoppola 1983, Craig 1983):
$V(n) = \tan(\pi\alpha n + \theta)$

Ballistic Diffusion in One-Dimensional Lattice-2

Theorem 2

$p-$periodic, $p \in \mathbb{Z}_+$: $V_{n+p} = V_n$, $\forall n \in \mathbb{Z}$ There exists a constant $0 < C < \infty$, depending on ϕ_0 and $\{V_n\}_{n=1}^{p}$, such that

$$\lim_{t \to \infty} \frac{\sqrt{m_2(t)}}{t} = C$$

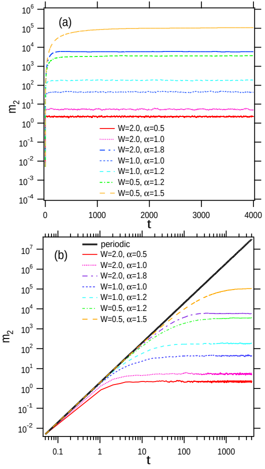
- Purely absolutely continuous spectrum $\to m_2 \sim t^2$ Ballistic motion?

•Note: Anomalous diffusion, $< x^2 > \sim t^{\alpha} (0 < \alpha < 2)$, not denied even if it is pure point spectrum

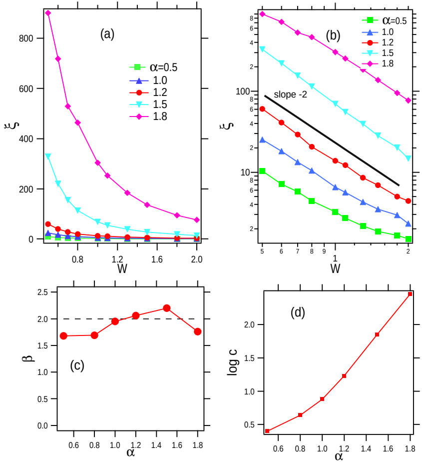
- Dynamical localization length (DLL):

$$\xi_{msd} = \sqrt{m_2(t \to \infty)}$$

- DLL $\xi_{msd} = \sqrt{m_2(t \to \infty)}$



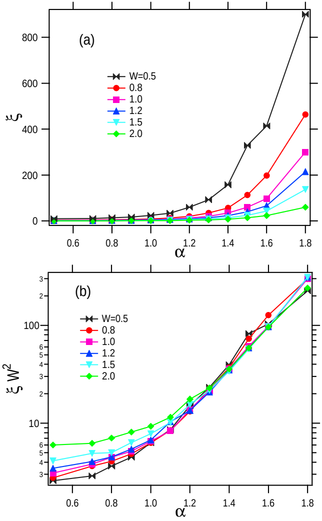
- MSD



$\alpha/W-$dependence

quantum diffusion-2(FFM)

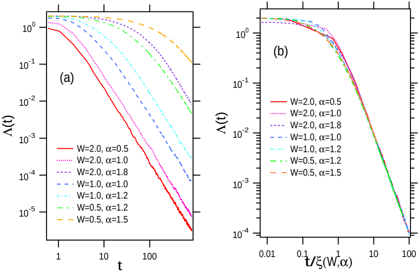
• Scaling of DLL



$\alpha-$**dpendence**

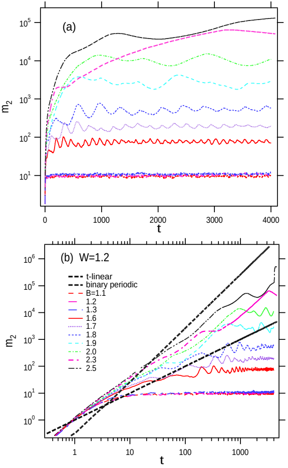
• Scaling of normalized MSD

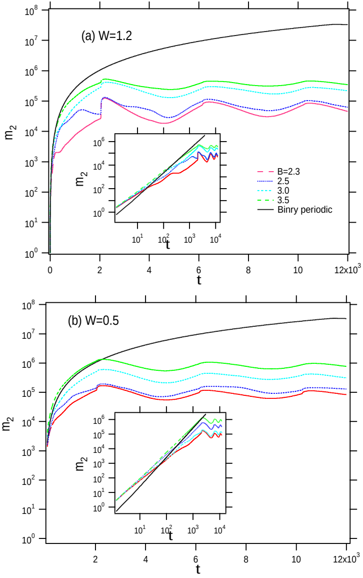
$$\Lambda(t,\alpha,W) = \frac{m_2(t)}{t^2} = F\left(\frac{t}{\xi_{msd}(\alpha,W)}\right),$$



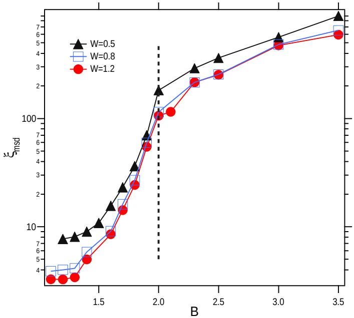
well-scaled

- MSD of MB system

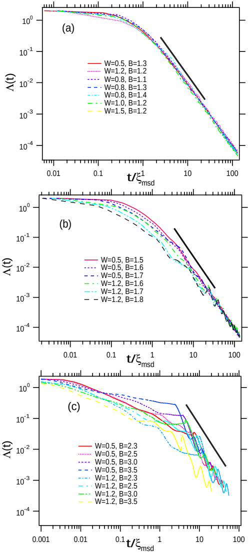




- DLL and scaling



$B-$dependence changes at $B = 2(\alpha = 1)$



Scaling of normalized MSD