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Analyticity property of wavefunction in complexifing the argument: critical state of Harper model

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OUTLINE

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(1) Motivation

Quantum dynamics of classically chaotic system coupled with a system with few degrees of freedom shows dissipative behaviors, i.e. stationary energy transport, normal diffusion of wavepacket, and so on.
K.S. Ikeda, Ann. Phys. 227, 1(1993)
H.Yamada and K.S.Ikesa, PRE65, 046211(2002).

• How does the feature of the wavefunction connected with the dissipation?

⟹ Analyticity property of wavefunction in complexifing the arguments

Analogy to breakdown of KAM curves in for Hamiltonian map systems, i.e. analytic domains of Lindstedt series in standard map, Henon map, and so on.
J.M.Greene and I.C. Percival, Physica D 3, 530(1981)
A.Berretti and L.Chierchia, Nonliniarity 3, 39(1990)

cf. Generally, the spectra of the operator are categorized into three cases; absolute continuous spectrum, singular continuous spectrum and pure points spectrum. However, the continuous spectrum does not always characterize the dissipative phenpmena as seen in Bloch states.

(2) Pade Approximation

- **Definition**

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$f(x) \sim \frac{a_0 + a_1 x + a_2 x^2 + \ldots . a_L x^L}{1 + b_1 x + b_2 x^2 + \ldots\ldots . b_M x^M} = \frac{P_L(x)}{Q_M(x)} \equiv [L|M]$$

$$f(x) - [L|M] = O(x^{L+M+1})$$

- **Calculation**

$$\sum_{j=1}^{M} c_{N+i-j} b_j = -c_{N+i} (i = 1, ..., M), \sum_{j=0}^{n} c_{n-j} b_j = a_n (n = 0, ..., N)$$

$$\begin{pmatrix} c_N & c_{N-1} & \cdots\cdots & \cdots\cdots & c_{N+1-M} \\ c_{N+1} & c_N & \cdots\cdots & \cdots\cdots & c_{N+2-M} \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ c_{N+M-1} & c_{N+M-2} & \cdots\cdots & \cdots\cdots & c_N \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_M \end{pmatrix} = - \begin{pmatrix} c_{N+1} \\ c_{N+2} \\ \cdots \\ \cdots \\ c_{N+M} \end{pmatrix}$$

Toeplitz matrix

- **Diagonal Pade App.** $L = M$

  Continued fraction function:

$$f(x) = c_0 + \cfrac{\alpha_1 x}{1 + \cfrac{\alpha_2 x}{1 + \ldots\ldots}}$$

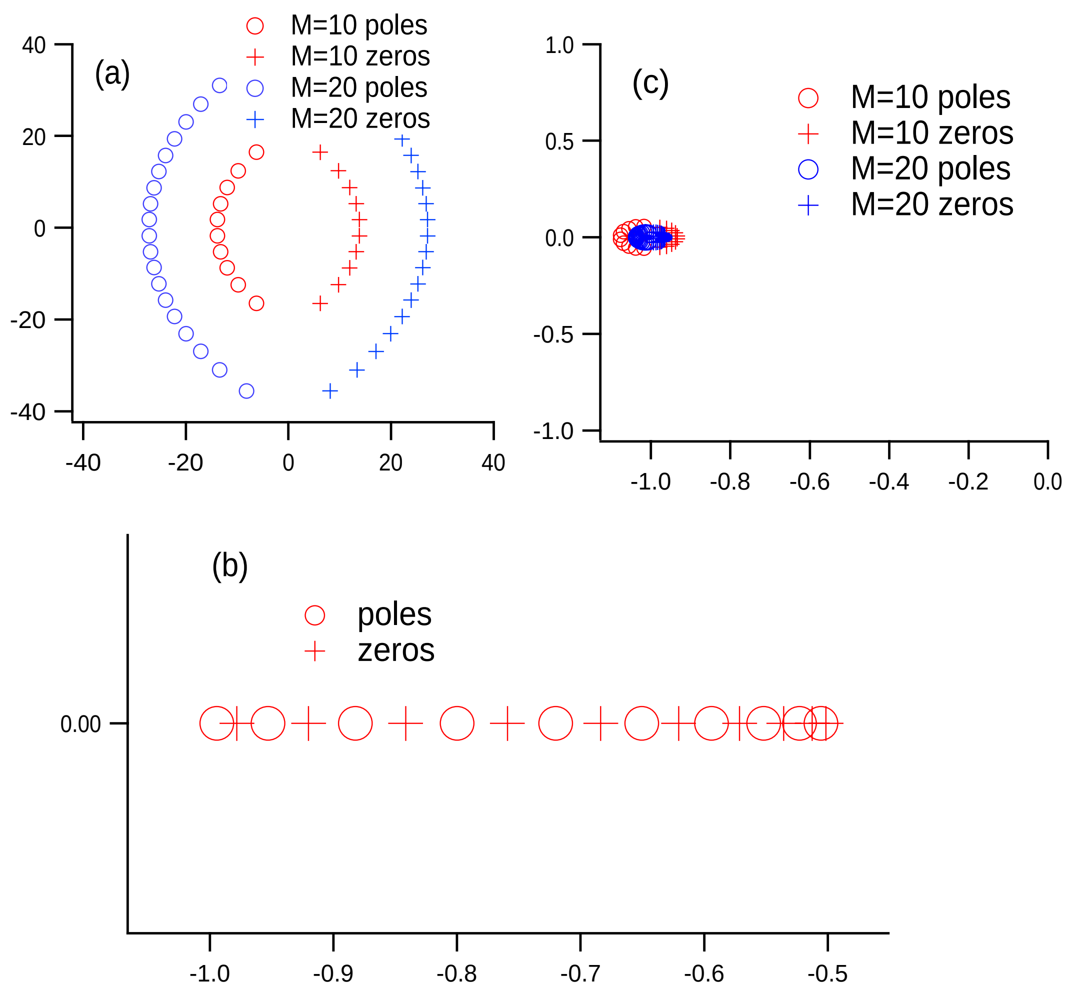
$2N$ order rational App. $\Leftrightarrow$ $[N|N]$ Pade App.

- **Notices**
  - The singularity of the function $f(x)$ is approximated by zeros of $[M|M]$.
  - Zeros($P_L(x) = 0$) and poles ($Q_M(x) = 0$) are sometime cancelled

George A. Baker, *Essentials of Pade Approximants* (Academic Press, 1975)
George A. Jr Baker and Peter Graves-Morris, *Pade Approximants 2nd edition*, (Cambridge University Press, 1996)

Examples of Pade App. for some functions with singularity (poles and brunch cut)

- (a) $f(x) = e^{-x}$ (no singularity)
  The poles and zeros go infinity as $M \to \infty$.
- (b) $f(x) = \sqrt{\frac{1+2x}{1+1}}$ (brunch cut)
  Pade App. clusters along the cut alternately with pole and zero.
- (c) $f(x) = e^{-x/(1+x)}$ (essencial singularity at $x = -1$)
  Pade App. clusters poles and zeros at the singular point.

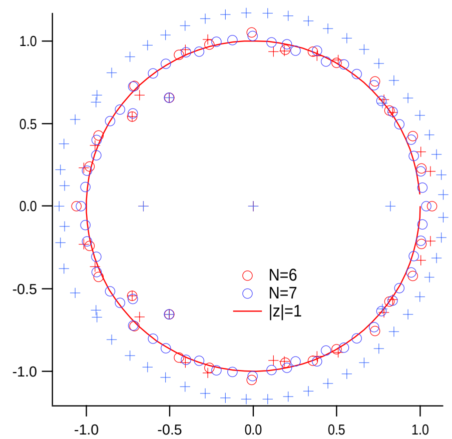


Examples: natural boundary at $|z| = 1$

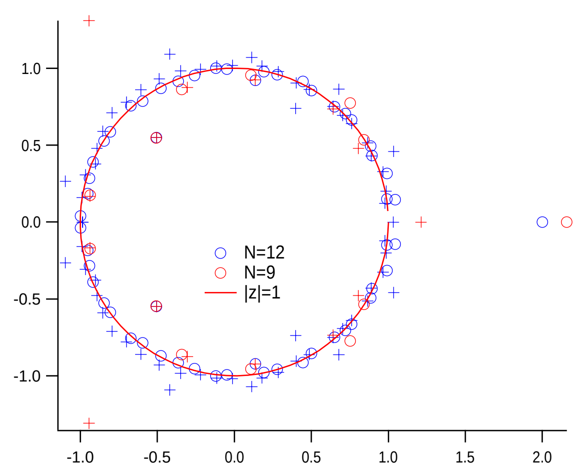
- $f_N(x) = \Sigma_{n=0}^{N} x^{2^n} \to f(x)$

$$f(x) \sim [N|N] = \frac{x + 2\sum_{k=2} x^{2^{n-1}} - 2x^{2^{N-2}}}{1 + \Sigma_{k=0}^{N-2} x^{2^k} - x^{2^{N-1}}}$$

$[2N-1|2N-1]$ Pade App.
$1 + \Sigma_{k=0}^{N-2} x^{2^k} - x^{2^{N-1}} = 0 \Rightarrow$ poles



- $f_N(x) = \Sigma_{n=0}^{N} x^{F_n} \to f(x)$, F_n : nth Fibonacci number
$1 + x^{F_{N-4}} - x^{F_{N-2}} = 0 \Rightarrow$ poles



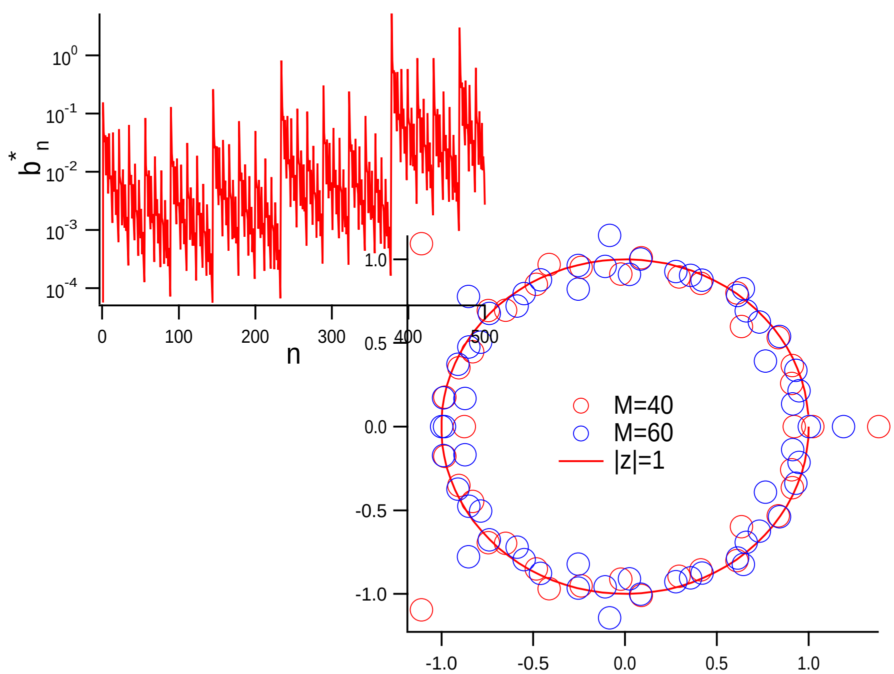
Pade App. for invariant curve (semistandard map)

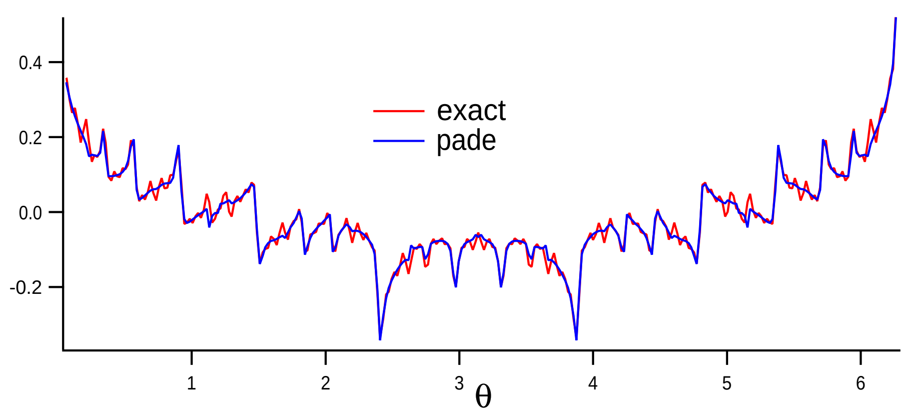
(Berretti, et al, 1990)

$$\Theta_{n+1} + \Theta_{n-1} - 2\Theta_n + ksin(\Theta_n) = 0$$

$$q(\theta + \omega) + q(\theta - \omega) - 2q(\theta) = ik\exp\{iq(\theta)\}$$

$$\begin{aligned} g(\theta) &= \sum_{n=1} b_n x^n, x = k\exp\{i\theta\} \\ &= \sum_{n=1} b_n^*(x/r)^n, b_n^* = r^n b_n(\text{scaled coeff.}) \end{aligned}$$





(3) Harper Model and Critical States

$$\Psi(n-1) + \Psi(n+1) + 2V\cos(2\pi\alpha n + \phi_0)\Psi(n) = E\Psi(n)$$

α: irrational number (golden mean)
V: potential strength
ϕ_0: initial phase (=0 for simplicity)

- **Aubry self-duality**

$$\Psi(P) \propto \sum_{n=1}^{N} exp\{-i(2\pi\alpha n + \phi_0)P\}\Psi(n)$$

$$2\cos(2\pi\alpha P + \phi_0)\Psi(P) + V(\Psi(P+1) + \Psi(P-1)) = E\Psi(P)$$

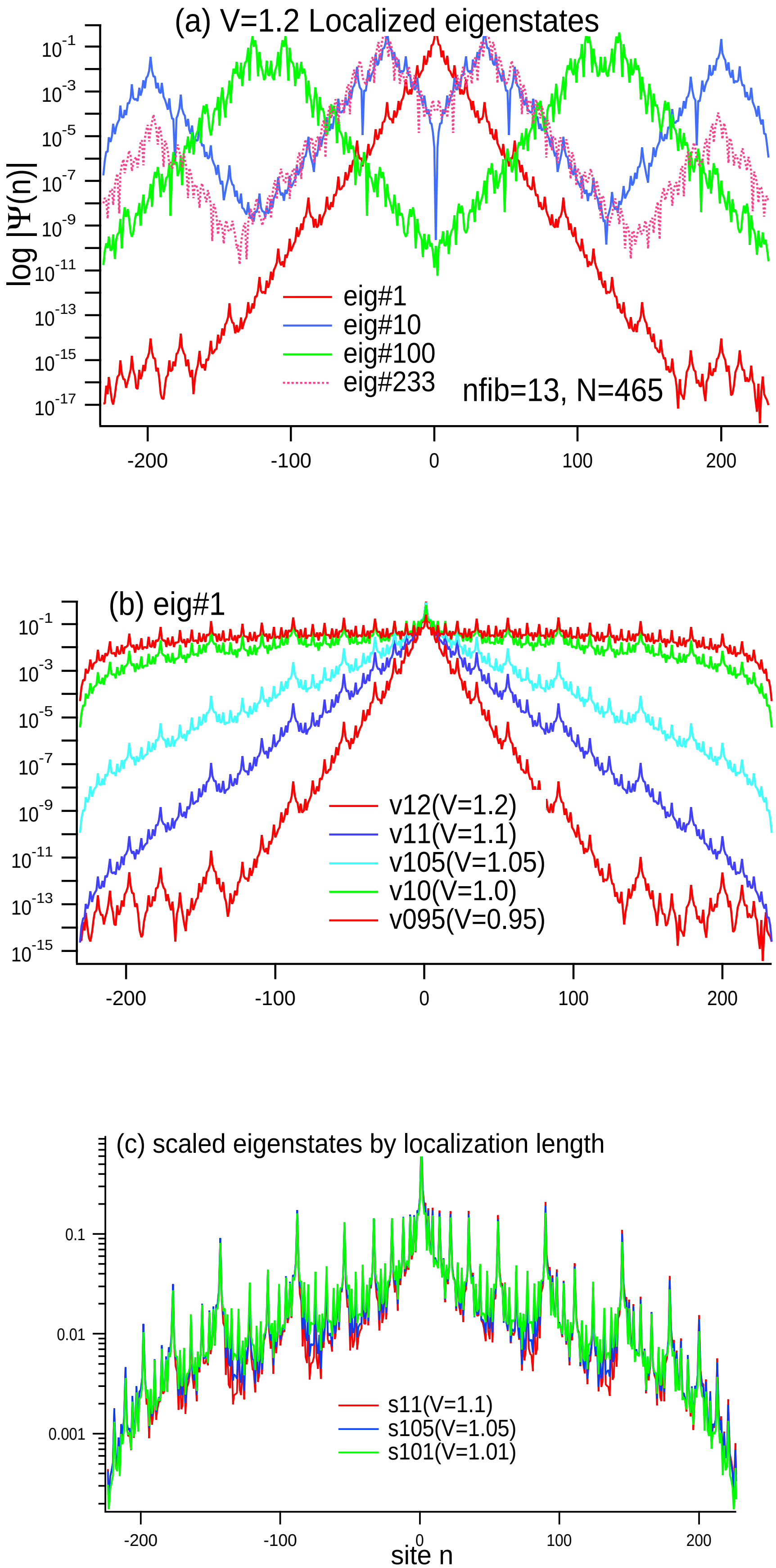
- **Spectral property**

$$S(\alpha, V) = \frac{V}{2}S(\alpha, \frac{2}{V}), \lim_{q\to\infty} |S(p/q, V)| = 4|1 - |V||$$

- **Metal-Insulator transition at α =Diophantine No.**
  $V > 1 \Longrightarrow$ exponentially localized states,
  (localization length $\xi = 1/\ln|V|$)
  $V = V_c = 1 \Longrightarrow$ critical states
  $V < 1 \Longrightarrow$ extended states

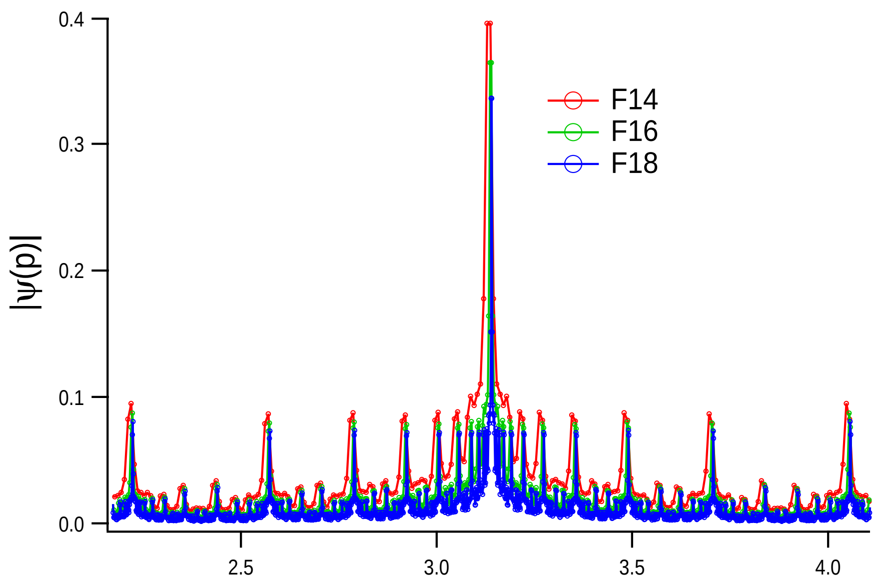
- **Diffusion of initially localized wavepacket**
  At $V = 1, < (\Delta x)^2 >\sim t^{\alpha}, \alpha = 0.97$ (Hiramoto and Abe88, Wilkinson94)

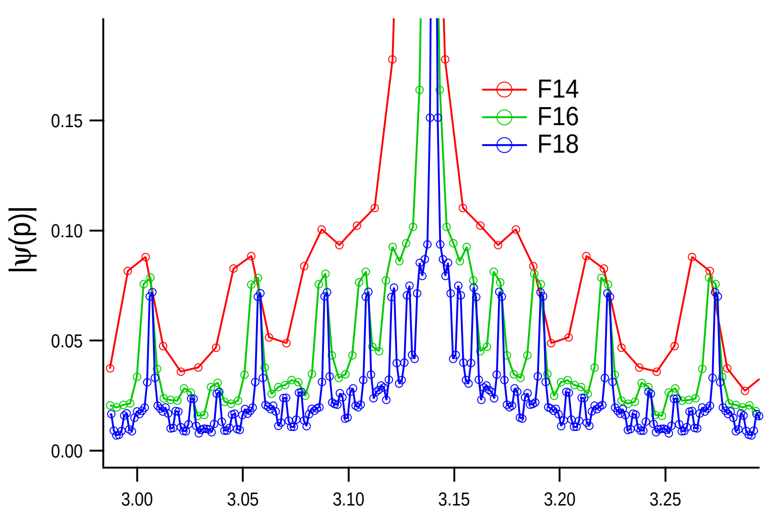
Eigenstates of Harper model



Fourier representation of eigenstates

$$\Psi(p) = \sum_n^N e^{ipn}\Psi(n), p = 2\pi/N, N = 2F_n - 1, \alpha = \frac{F_{n-1}}{F_n}$$





c.f. Analogus to complex torus for the standard map

Pade approximation for quantum states

$$\psi(z) = \psi_+(z) + \psi_-(z)$$

singularity analytic

$$\begin{aligned}
\psi_+(p) \Longrightarrow \psi_+(p+iq) &\propto \sum_{n=0}^{N_r} \exp\{-izn\}\Psi(n) \\
z &= \zeta + i\eta = \frac{2\pi}{N}p + i\frac{2\pi}{N}q \\
\eta_c &: \text{ critical depth of the analytic domain}
\end{aligned}$$

Pade App. for some states 1(localized states)

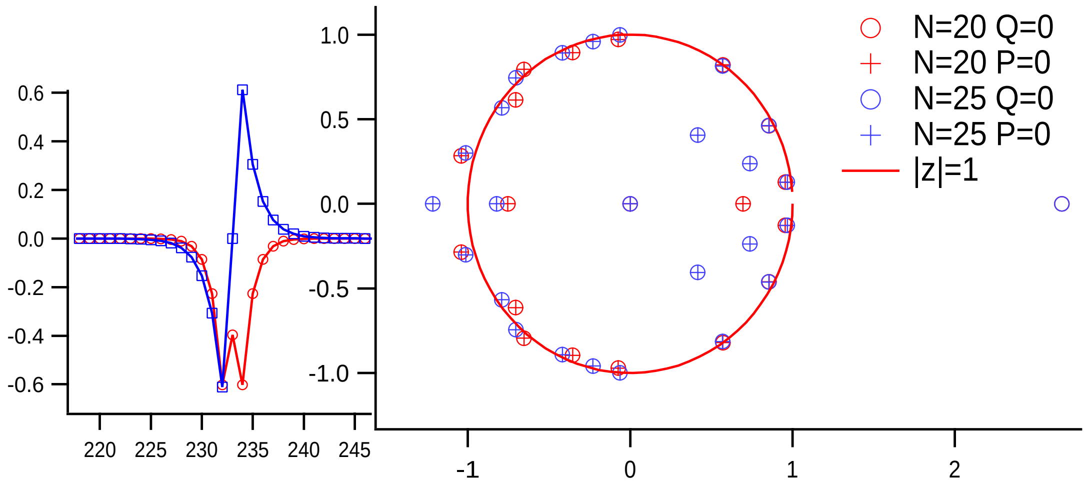


Figure 1: **Impurity localized state**

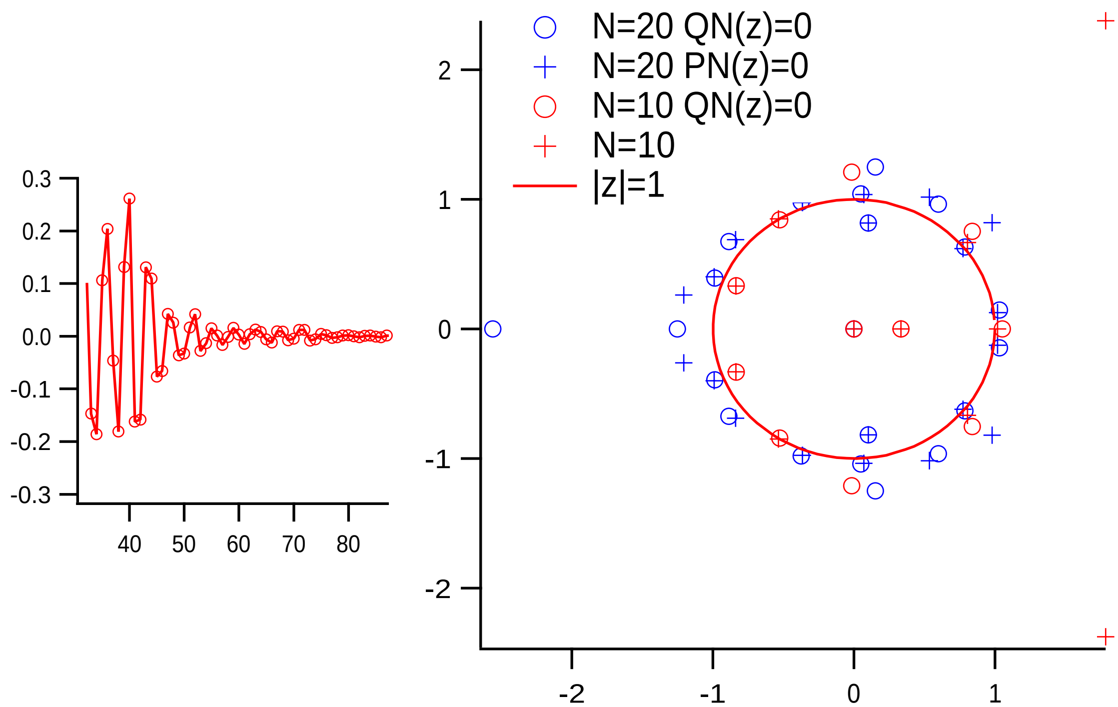
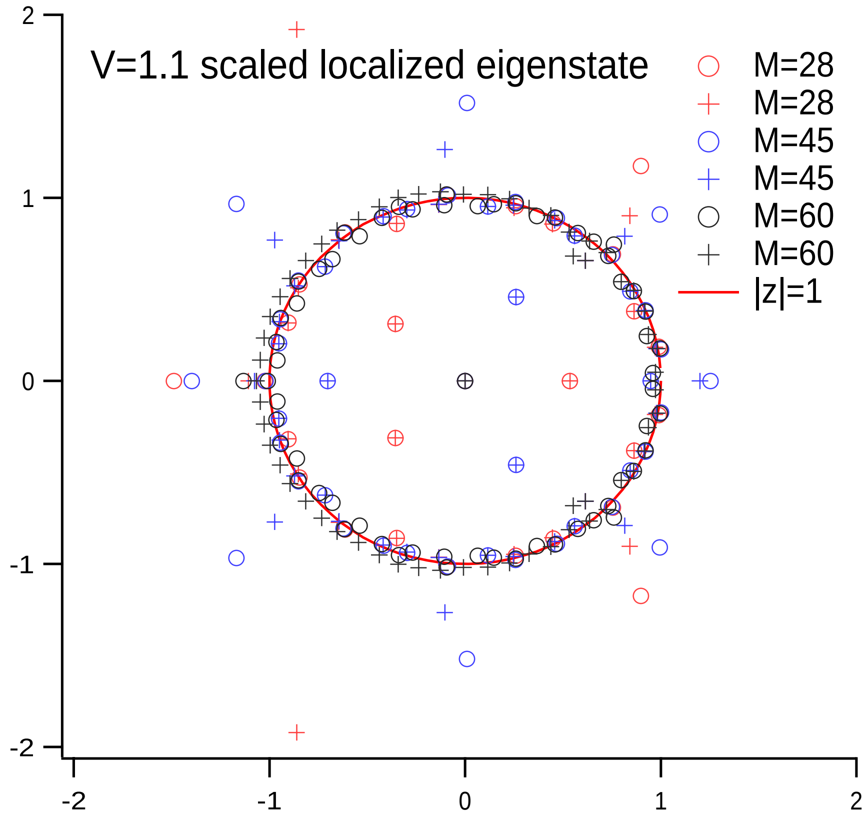
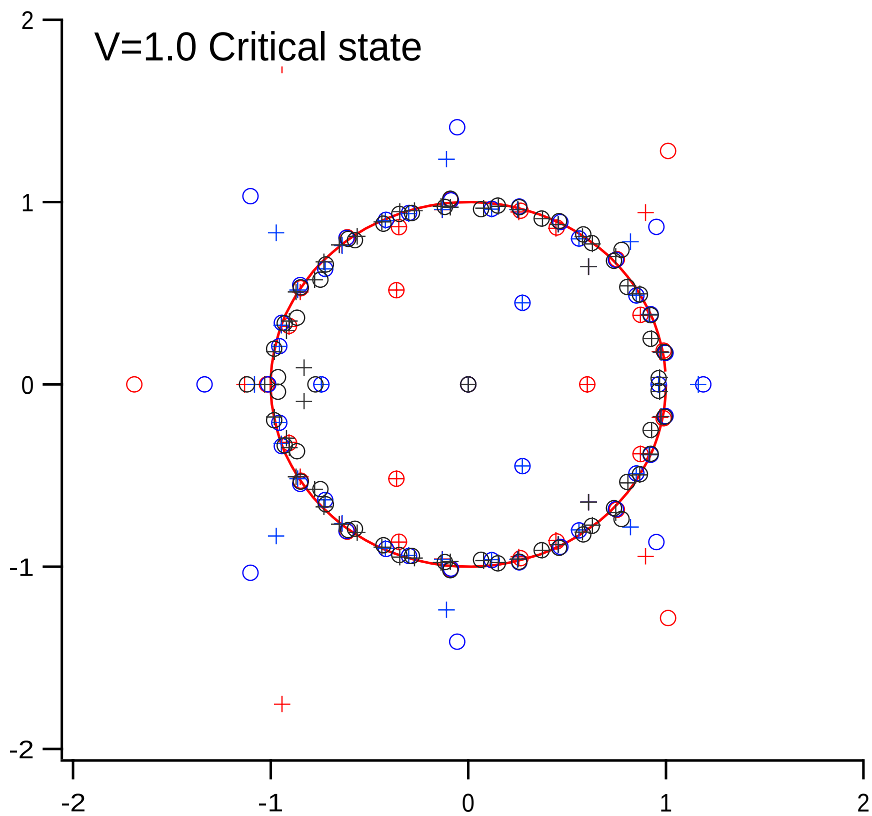


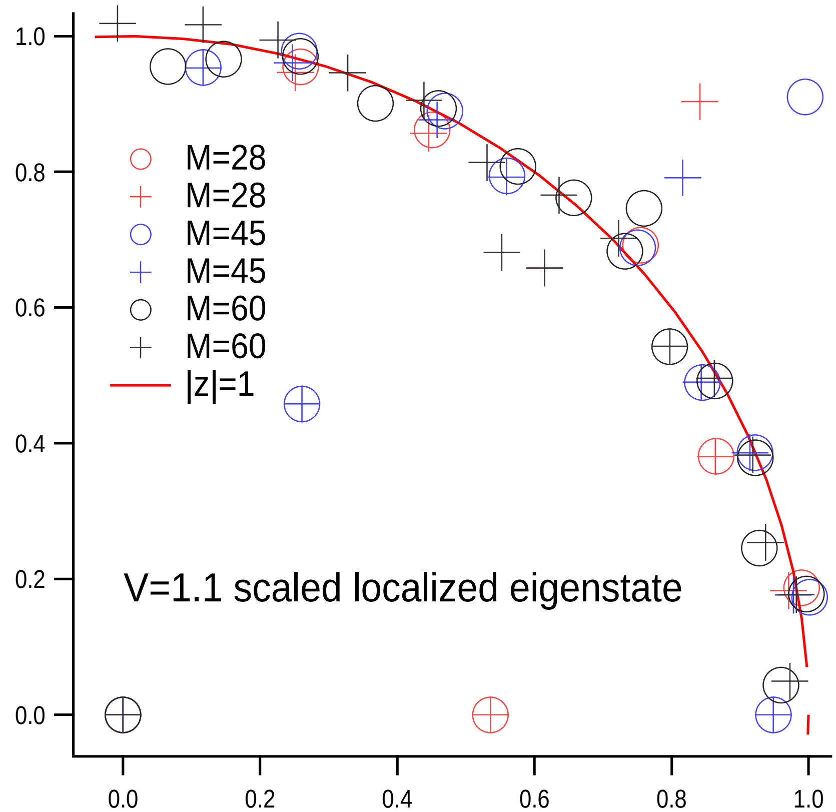
Figure 2: **Anderson localized state**

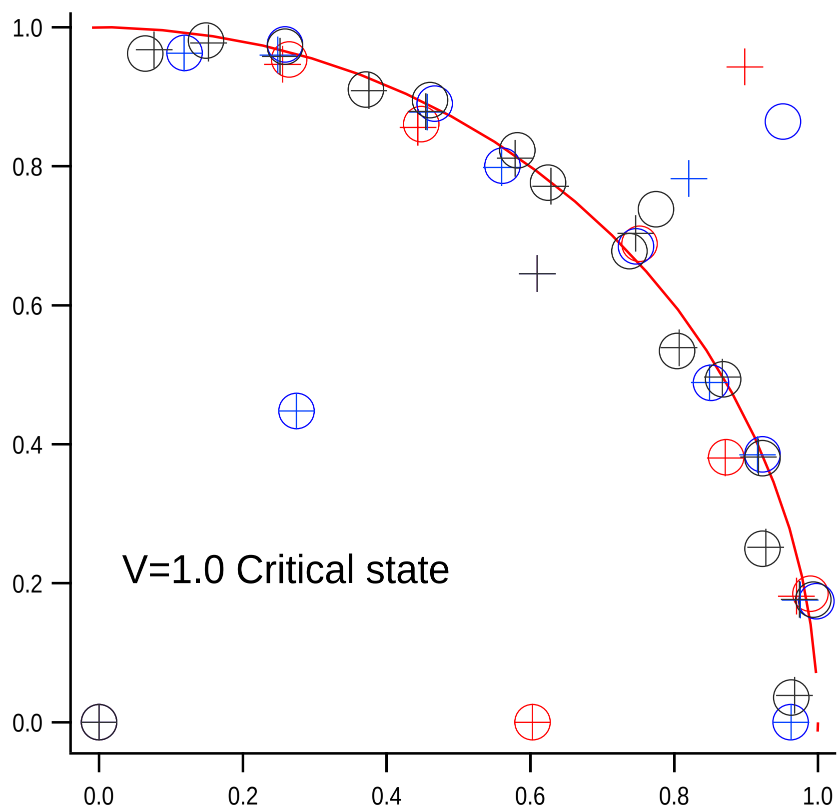
Pade App. for some states 2 (Harper model)





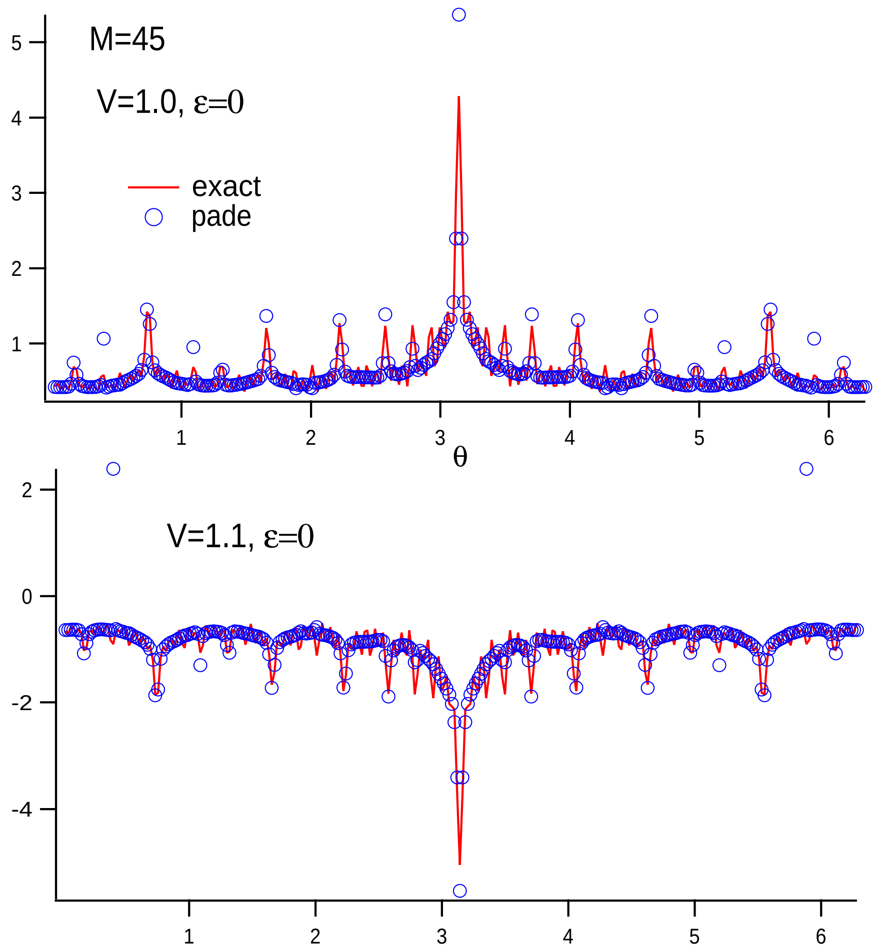
Extension of Pade App. for some states 2

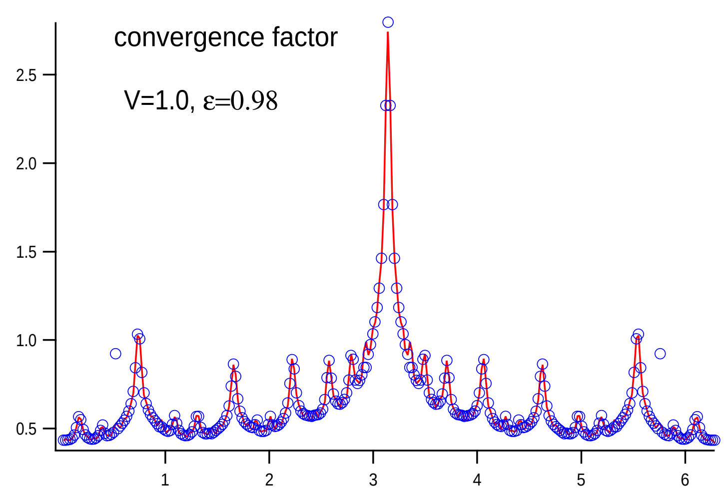




Check for Pade App.

$$\Psi(\theta) = \sum_{n}^{N} (\epsilon e^{i\theta})^n \Psi(n), \epsilon : \text{convergence factor}$$





Direct Method for Singularity test

How kinds of the singularity does a function $f(z)$ with convergence radius $|z| = 1$ have? poles or natural boundary?

Expansion near the origin:

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

Expansion near the edge of the convergence domain $w = (1-\epsilon)e^{i\theta}$:

$$f(z) = \sum_{n=0} b_n (z-w)^n$$

We can get the coefficients $\{b_m\}$ as

$$b_m = \sum_{n=m}^{N} n(n-1)...(n-m+1)w^{n-m}a_n/m!$$
$$N = 2n_0 \sim 2m/|\log(1-\epsilon)|$$

Then the convergence radius : $r(w) = \lim_{n\to\infty} |b_n|^{-1/n}$
If $r(\omega) = \epsilon$ **for any** $\theta \Rightarrow |z| = 1$ **is natural boundary**

We have confirmed that $|z| = 1$ is natural boundary for the critical state by the direct method.

(5) Summary and Discussion

The singularity of the states as $N \to \infty$

- Impurity states $\Leftrightarrow$ Simple poles
- Localized states $\Leftrightarrow$ Natural boundary
- Critical states $\Leftrightarrow$ Natural boundary

Conjecture:

- The dissipative or pre-dissipative states might be characterized by **natural boundary**.
- Necessary conditions for dissipative phenomena in closed quantum system are at least **continuous spectrum**+ **natural boundary**.

c.f. $N \leq 70$ in our numerical calculation for Pade approximation